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Abstract In this work we deal with the exponential stability of the nonlinear Korteweg-de Vries (KdV) equation on a finite star-shaped network in the presence of delayed internal feedback. We start by proving the well-posedness of the system and some regularity results. Then we state an exponential stabilization result using a Lyapunov function by imposing small initial data and a restriction over the lengths. In this part also, we are able to obtain an explicit expression for the rate of decay. Then we prove the exponential stability of the solutions without restriction on the lengths and for small initial data, this result is based on an observability inequality. After that, we obtain a semi-global stabilization result working directly with the nonlinear system. Next we study the case where it may happen that a control domain with delay is outside of the control domain without delay. In that case, we obtain also a local exponential stabilization result. Finally, we present some numerical simulations in order to illustrate the stabilization.

Keywords KdV equation · delay · stabilization · Star-Shaped Network

Mathematics Subject Classification (2010) 35Q53 · 35B35 · 35R02

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1 Introduction

The Korteweg-de Vries (KdV) equation $u_t + u_x + u_{xxx} + uu_x = 0$ was introduced in [8] to model the propagation of long water waves in a channel. It has been widely studied in the last years, in particular its controllability and stabilization properties, see [3, 19] for a complete introduction to those problems.
From the stabilization point of view, we can refer to the work [22] where the boundary exponential stabilization problem was studied in the bounded spatial domain $x \in (0,1)$. It is well known that the length L of the spatial domain plays an important role in the stabilization and controllability properties of the KdV equation. For example if $L = 2\pi$ it is possible to find a solution of the linearization around 0 of KdV ($u(t,x) = 1 - \cos(x)$) which has constant energy. More generally if $L \in \mathcal{N}$ where $\mathcal{N}$ is called the set of critical lengths defined by

$$\mathcal{N} = \left\{ 2\pi\sqrt{\frac{k^2 + kl + l^2}{3}},\ k, l \in \mathbb{N}^* \right\},$$

we can find suitable initial data such that the solution of the linear KdV equation has constant energy. In the case of internal stabilization it is proven in [17, 15] that for any critical length by adding a localized damping we reach the local exponential stability for the nonlinear KdV equation.

Adding a delay term allows to study the action of a device in a more real-life setting. It is known that even the presence of small delays in internal feedback could destabilize a system, see for example [6]. In the works [2] and [21] the problem of robustness with respect to time delay for a KdV equation was studied with boundary and internal stabilization respectively. Our contribution to this work is to study the stabilization of a KdV equation posed on a Star Network in presence of internal time delays. With respect to the KdV equation on Networks the first work introducing this system was [1] where the stabilization and controllability problems were studied and after that the boundary controllabilty results were improved in [4].

In this work we are interested in the stability properties of the Korteweg-de Vries equation with internal input delay posed on a Star-Shaped Network. Let $K = \{k_j : 1 \leq j \leq N\}$ be the set of the edges of a network $\mathcal{T}$ described as the intervals $[0, \ell_j]$ with $\ell_j > 0$ for $j = 1, \cdots N$, the network $\mathcal{T}$ is defined by

$$\mathcal{T} = \bigcup_{j=1}^{N} k_j.$$

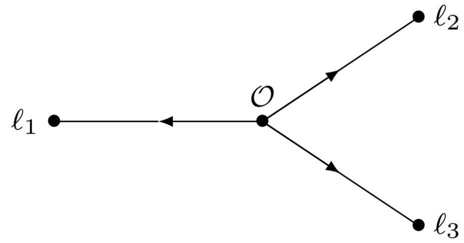

Fig. 1 Star Shaped Network for $N = 3$.

Specifically we are going to consider the next evolution problem for the KdV equation with internal input delay on each edge.

$$\begin{cases} \partial_t u_j(t,x)+\partial_x u_j(t,x)+u_j(t,x)\partial_x u_j(t,x)+\partial_x^3 u_j(t,x) & \\ +a_j(x)u_j(t,x)+b_j(x)u_j(t-h_j,x)=0, & x\in(0,\ell_j),\ t>0,\ j=1,\cdots,N, \\ u_j(t,0)=u_k(t,0), & \forall j,k=1,\cdots N, \\ \sum_{j=1}^N \partial_x^2 u_j(t,0)=-\alpha u_1(t,0)-\frac{N}{3}u_1^2(t,0), & t>0, \\ u_j(t,\ell_j)=\partial_x u_j(t,\ell_j)=0, & t>0,\ j=1,\cdots,N, \\ u_j(0,x)=u_j^0(x), & x\in(0,\ell_j), \\ u_j(t,x)=z_j^0(t,x), & (t,x)\in(-h_j,0)\times(0,\ell_j), \end{cases} \tag{KdVd}$$

where $\alpha \geq \frac{N}{2}$ and for all $j = 1,\cdots,N$, $h_j > 0$ is the time delay on the edge j, a_j, $b_j \in L^\infty(0,\ell_j)$ are non-negative and supp $b_j = \omega_j$ is a nonempty, open subset of $(0,\ell_j)$ such that

$$b_j(x) \geq b_0 > 0, \text{ a.e on } \omega_j, \tag{1.1}$$

$$\text{there exists } c_0 > 0, \text{ such that } b_j(x)+c_0 \leq a_j(x), \forall x \in \omega_j. \tag{1.2}$$

The condition $\alpha > \frac{N}{2}$ was firstly introduced in [1] in order to have a decreasing energy, the case $\alpha = \frac{N}{2}$ was studied in [4] from a controllability point of view. In our work we consider $\alpha > \frac{N}{2}$ in some cases and $\alpha \geq \frac{N}{2}$ in others. The conditions over the damped terms with and without delay (1.1)-(1.2) are the analogues of the conditions $(1.2)-(1.3)$ presented in [21], similar conditions over the weight of the feedback with and without delay can be founded in [2, 13, 12].

In order to study this system we need first a proper functional setting. We define the following spaces

$$H_r^s(0,\ell_j) = \left\{ v \in H^s(0,\ell_j),\ \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{i-1} v(\ell_j) = 0,\ 1 \leq i \leq s \right\},\ s = 1,2,$$

where the index r is related to the null right boundary conditions. The space $\mathbb{H}_e^s(\mathcal{T})$ will be the cartesian product of $H_r^s(0,\ell_j)$ including the continuity condition on the central node $(u_j(0) = u_k(0), \forall j,k = 1,\cdots,N)$

$$\mathbb{H}_e^s(\mathcal{T}) = \left\{ \underline{u} = (u_1,\cdots,u_N) \in \prod_{j=1}^N H_r^s(0,\ell_j), u_j(0) = u_k(0), \forall j,k = 1,\cdots,N \right\}, s = 1,2$$

and

$$\|\underline{u}\|^2_{\mathbb{H}^1_e(\mathcal{T})} = \sum_{j=1}^{N} \|u_j\|^2_{H^1(0,\ell_j)}$$

where the index e indicates that each edge belongs to $H^s_r(0,\ell_j)$.

$$\mathbb{L}^2(\mathcal{T}) = \prod_{j=1}^{N} L^2(0,\ell_j), \qquad \mathbb{L}^\infty(\mathcal{T}) = \prod_{j=1}^{N} L^\infty(0,\ell_j).$$

The space $\mathbb{L}^2(\mathcal{T})$ is equipped with the inner product

$$(\underline{u},\underline{v})_{\mathbb{L}^2(\mathcal{T})} = \sum_{j=1}^{N} \int_0^{\ell_j} u_j v_j dx, \quad \forall \underline{u},\underline{v} \in \mathbb{L}^2(\mathcal{T}). \tag{1.3}$$

We also define the space

$$\mathbb{B} = C([0,T],\mathbb{L}^2(\mathcal{T})) \cap L^2(0,T;\mathbb{H}^1_e(\mathcal{T}))$$

endowed with the norm

$$\|u\|_{\mathbb{B}} = \|\underline{u}\|_{C([0,T],\mathbb{L}^2(\mathcal{T}))} + \|\underline{u}\|_{L^2(0,T;\mathbb{H}^1_e(\mathcal{T}))} = \max_{t\in[0,T]} \|\underline{u}\|_{\mathbb{L}^2(\mathcal{T})} + \left(\int_0^T \|\underline{u}(t,\cdot)\|^2_{\mathbb{H}^1_e(\mathcal{T})} dt\right)^{1/2}.$$

Note first (1.1) and (1.2) imply

$$\omega_j = \text{supp } b_j \subset \text{ supp } a_j, \text{ and } a_j(x) \geq b_0 + c_0 > 0, \text{ in } \omega_j. \tag{1.4}$$

To deal with delays we introduce the following space

$$\mathcal{H} = \mathbb{L}^2(\mathcal{T}) \times \left(\prod_{j=1}^{N} L^2((-h_j,0)\times(0,\ell_j))\right)$$

endowed with

$$\|(\underline{u},\underline{z})\|^2_{\mathcal{H}} = \sum_{j=1}^{N} \left(\int_0^{\ell_j} u_j(x)^2 dx + \int_{-h_j}^{0}\int_0^{\ell_j} \xi_j(x) z_j^2(s,x) dx ds\right)$$

where for all $j = 1,\cdots,N$, ξ_j is a non-negative function belonging to $L^\infty(0,\ell_j)$ such that supp $\xi_j =$ supp $b_j = \omega_j$ and

$$b_j(x) + c_0 \leq \xi_j(x) \leq 2a_j(x) - b_j(x) - c_0, \text{ in } \omega_j. \tag{1.5}$$

For (KdVd), we define the energy

$$E(t) = \sum_{j=1}^{N} \left(\int_0^{\ell_j} u_j^2(t,x) dx + h_j \int_{\omega_j}\int_0^1 \xi_j(x) u_j^2(t-h_j\rho,x) d\rho dx\right). \tag{1.6}$$

The above expression corresponds to the square norm of $(\underline{u}(t,\cdot),\underline{u}(t+\cdot,\cdot))$ in $\mathcal{H}$, with the change of variable $s = -h_j\rho$ for $u_j(t+s,x)$.

Finally we denote $\mathbb{L}^2(\Omega) = \prod_{j=1}^N L^2((0,1)\times\omega_j)$, and let

$$H = \mathbb{L}^2(\mathcal{T})\times\left(\prod_{j=1}^N L^2((0,1)\times\omega_j)\right) = \mathbb{L}^2(\mathcal{T})\times\mathbb{L}^2(\Omega)$$

with its inner product

$$\left\langle\left(\frac{\underline{u}}{\underline{z}}\right),\left(\frac{\underline{v}}{\underline{y}}\right)\right\rangle = \sum_{j=1}^N \int_0^{\ell_j} u_j(x)v_j(x)dx + h_j\int_{\omega_j}\int_0^1 \xi_j(x)z_j(\rho,x)y_j(\rho,x)d\rho dx,$$

we denote by $\|\cdot\|_H$ its associated norm.

Our first main result is the following one, where local exponential stability of (KdVd) is obtained for a restricted assumption over $L = \max_{j=1,\cdots,N}\ell_j$, but an estimation of the decay rate is given.

Theorem 1.1 *Assume $\underline{a}$, $\underline{b}\in\mathbb{L}^\infty(\mathcal{T})$ componentwise non-negative that satisfy* (1.1) *and* (1.2). *Let $\alpha > \frac{N}{2}$ and $(\ell_j)_{j=1}^N \subset (0,\infty)$ such that $L < \frac{\sqrt{3}}{2}\pi$. Then there exists $\epsilon > 0$, such that for every $(\underline{u}^0, \underline{z}^0(-\underline{h}\cdot,\cdot))\in H$ satisfying $\|(\underline{u}^0, \underline{z}^0(-\underline{h}\cdot,\cdot))\|_H \leq \epsilon$, the energy of* (KdVd) *defined by* (1.6) *decays exponentially. That is, there exist $C>0$, $\gamma>0$ such that*

$$E(t) \leq CE(0)e^{-2\gamma t}, \quad t>0,$$

where

$$\gamma \leq \min\left\{\frac{\left(3\mu_1\pi^2 + \frac{2}{3}L^{3/2}\epsilon\mu_1\pi^2 - \mu_1 4L^2\right)}{8L^2((1+L\mu_1))}, \min_{j=1,\cdots,N}\frac{\mu_2}{2h_j(\mu_2 + \|\xi_j\|_{L^\infty(0,\ell_j)})}\right\}, \tag{1.7}$$

$$C = \left(1 + \max\left\{L\mu_1, \frac{\mu_2}{b_0}\right\}\right),$$

for μ_1 and μ_2 such that

$$0 < \mu_1 < \min\left\{1, \frac{1}{N}(2\alpha - N)\min_{j=1,\cdots,N}\left\{\inf_{\omega_j}\frac{2a_j - b_j - \xi_j}{Lb_j}, \inf_{\omega_j}\frac{\xi_j - b_j}{Lb_j}\right\}\right\},$$

$$0 < \mu_2 < \min_{j=1,\cdots,N}\inf_{\omega_j}\left\{2a_j - b_j - \xi_j - \mu_1 Lb_j\right\}.$$

This result will be proved in a constructive way by using a Lyapunov function, similar to those used in [2, 21].

On the other hand, our second main result is obtained without restriction on the lengths of $\mathcal{T}$ and gives us a local exponential stability.

Theorem 1.2 *Assume $\underline{a}$, $\underline{b} \in \mathbb{L}^\infty(\mathcal{T})$ componentwise non-negative that satisfy* (1.1) *and* (1.2). *Let $(\ell_j)_{j=1}^N \subset (0,\infty)$, then there exists $\epsilon > 0$ such that for all $(\underline{u}^0, \underline{z}^0(-\underline{h}\cdot,\cdot)) \in H$ with $\|(\underline{u}^0, \underline{z}^0(-\underline{h}\cdot,\cdot))\|_H \leq \epsilon$ the energy of* (KdVd) *decays exponentially, i.e, there exists $C > 0$ and $\mu > 0$ such that $E(t) \leq CE(0)e^{-\mu t}$ for all $t > 0$.*

The main difference between Theorem 1.1 and Theorem 1.2 is that Theorem 1.2 is based on an observability inequality which is proved using a contradiction argument. Thus we can not estimate the decay rate.

In our third main result, we proved a semi-global exponential stabilization by working directly with the nonlinear system (KdVd).

Theorem 1.3 *Assume $\underline{a}$, $\underline{b} \in \mathbb{L}^\infty(\mathcal{T})$ componentwise non-negative that satisfies* (1.1) *and* (1.2). *Let $(\ell_j)_{j=1}^N \subset (0,\infty)$ and $R > 0$. Then for all $(\underline{u}^0, \underline{z}^0(-\underline{h}\cdot,\cdot)) \in H$ with $\|(\underline{u}^0, \underline{z}^0(-\underline{h}\cdot,\cdot))\|_H \leq R$ then there exist $C = C(R) > 0$ and $\mu = \mu(R) > 0$ such that the energy of* (KdVd) *satisfies $E(t) \leq CE(0)e^{-\mu t}$ for all $t > 0$.*

The semi-global sense of this result arises from the fact that we can choose as we want the parameter $R > 0$ of the initial data but the decay rate depends on it.

In the last results presented, it is not possible to take $a_j = 0$ and $b_j \neq 0$ for some $j \in \{1, \cdots, N\}$ (by (1.4) if $a_j = 0$ then $b_j = 0$). However this is only a technical part of the proof and in the next result we deal with this problem in a more general case, we suppose for this part that

$$\omega_j = \operatorname{supp} b_j \not\subset \operatorname{supp} a_j, \quad \text{for } j \in I \subset \{1, \cdots, N\}. \tag{1.8}$$

For this we write now the analogues of the condition (1.2) in the setting (1.8), take $I^* = \{1, \cdots, N\} \backslash I$,

$$\text{there exists } c_0 > 0, \text{ such that } b_j(x) + c_0 \leq a_j(x), \forall x \in \omega_j, \quad \text{for } j \in I^*. \tag{1.9}$$

Then we write our last result of stabilization when the internal delay is not necessarily supported in the domain of a_j.

Theorem 1.4 *Assume $\underline{a}$, $\underline{b} \in \mathbb{L}^\infty(\mathcal{T})$ componentwise non-negative that satisfy* (1.1) *and* (1.9). *Let $\alpha > \frac{N}{2}$, $\eta > 1$ and $(\ell_j)_{j=1}^N \subset (0,+\infty)$ such that $L < \frac{\sqrt{3}}{2}\pi$. Then there exits $\delta = \delta(\alpha, \eta, L, \underline{h}) > 0$ and $\epsilon > 0$, such that for every $(\underline{u}^0, \underline{z}^0(-\underline{h}\cdot,\cdot)$ satisfying $\|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})} \leq \delta$ and $\|(\underline{u}^0, \underline{z}^0(-\underline{h}\cdot,\cdot))\|_H \leq \epsilon$, the energy of* (KdVd) *decays exponentially to* 0.

The organization of this paper is the following:

Section 2 is devoted to the study of the well-posedness of (KdVd). More precisely we consider the linearization around 0 of (KdVd) and using Semigroup Theory we show the well-posedness of the linear system. Then using a fixed point argument we obtain the well-posedness for the nonlinear system. In Section 3 we present our stabilization results when the feedback terms $\underline{a}, \underline{b} \in \mathbb{L}^\infty(\mathcal{T})$ satisfy (1.1) and (1.2). The first one namely Theorem 1.1 is obtained following the same steps as [2, 21]. Then we detail the proofs of Theorem 1.2 and Theorem 1.3 that are based on an observability inequality. In Section 4, we study the case where (1.2) is not satisfied and we show the proof of Theorem 1.4 using a suitable auxiliary system and a perturbation argument. Some numerical simulations are presented in Section 5 in order to illustrate the results obtained. Section 6 collects some concluding ideas and future research lines.

2 Well-posedness of a delayed KdV system

Our idea is the following, first we work with the linearization around 0 of (KdVd), then we add a boundary source term at the central node $g(t)$ to consider the nonlinear boundary condition $-\dfrac{N}{3}u_1^2(t,0)$and secondly we add the internal source terms f_j to consider after the term $u_j\partial_x u_j$. Finally to pass to the nonlinear (KdVd) we use a fixed point argument.

2.1 Well-posedness of the linear case

We start by proving the well-posedness for the linearization of (KdVd) around 0, that writes

$$\begin{cases} \partial_t u_j(t,x)+\partial_x u_j(t,x)+\partial_x^3 u_j(t,x)+a_j(x)u_j(t,x) & \\ +b_j(x)u_j(t-h_j,x)=0, & x\in(0,\ell_j),\ t>0,\ j=1,\cdots,N,\\ u_j(t,0)=u_k(t,0), & \forall j,k=1,\cdots N,\\ \sum_{j=1}^N \partial_x^2 u_j(t,0)=-\alpha u_1(t,0), & t>0,\\ u_j(t,\ell_j)=\partial_x u_j(t,\ell_j)=0, & t>0,\ j=1,\cdots,N,\\ u_j(0,x)=u_j^0(x), & x\in(0,\ell_j),\\ u_j(t,x)=z_j^0(t,x), & (t,x)\in(-h_j,0)\times(0,\ell_j). \end{cases} \tag{LKdVd}$$

We set $z_j(t,\rho,x)=u_j\big|_{\omega_j}(t-h_j\rho,x)$ $x\in\omega_j$, $\rho\in(0,1)$. Then we can check that

$$\begin{cases} h_j\partial_t z_j(t,\rho,x)+\partial_\rho z_j(t,\rho,x)=0, & x\in\omega_j,\rho\in(0,1),t>0,\\ z_j(t,0,x)=u_j(t,x), & x\in\omega_j,t>0,\\ z_j(0,\rho,x)=u_j\big|_{\omega_j}(-h_j\rho,x)=z_j^0(-h_j\rho,x), & \rho\in(0,1). \end{cases} \tag{2.1}$$

Let us introduce the componentwise product $.*$ as

$$\begin{pmatrix} p_1\\ \vdots\\ p_N\end{pmatrix} .* \begin{pmatrix} q_1\\ \vdots\\ q_N\end{pmatrix} = \begin{pmatrix} p_1q_1\\ \vdots\\ p_Nq_N\end{pmatrix}.$$

Then (LKdVd) can be written as

$$\begin{cases} U_t(t)=\mathcal{A}U(t),\ t>0\\ \quad U(0)=U_0, \end{cases} \tag{2.2}$$

where $U = \begin{pmatrix} \underline{u} \\ \underline{z} \end{pmatrix}$, $U_0 = \begin{pmatrix} \underline{u}^0 \\ \underline{z}^0\big|_{\omega}(-\underline{h}\cdot,\cdot) \end{pmatrix}$ and the operator $\mathcal{A}$ is defined by:

$$\mathcal{A}U = \begin{pmatrix} -(\mathrm{D}_x(\mathcal{T}) + \mathrm{D}_x^3(\mathcal{T}))\underline{u} - \underline{a}.*\underline{u} - \underline{b}.*\underline{\tilde{z}}(1,\cdot) \\ -\dfrac{1}{\underline{h}}.*\mathrm{D}_\rho(\mathcal{T})\underline{z} \end{pmatrix}$$

for $\underline{u} = (u_j)_{j=1}^N$, $\underline{a} = (a_j)_{j=1}^N$, $\underline{b} = (b_j)_{j=1}^N$, $\underline{h} = (h_j)_{j=1}^N$, $\left(\dfrac{1}{\underline{h}}\right)_j = \dfrac{1}{h_j}$ and $\underline{\tilde{z}}(1,\cdot) = (\tilde{z}_j(1,\cdot))_{j=1}^N$ in which $\tilde{z}_j(1,\cdot) \in L^2(0,\ell_j)$ is the extension by 0 of $z_j(1,\cdot)$ outside ω_j and the operators $\mathrm{D}_x(\mathcal{T})$ (resp. $\mathrm{D}_\rho(\mathcal{T})$) acts like the derivative with respect to x (resp ρ) componentwise as

$$\mathrm{D}_x(\mathcal{T})\begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} = \begin{pmatrix} \partial_x u_1 \\ \vdots \\ \partial_x u_N \end{pmatrix}, \quad \mathrm{D}_\rho(\mathcal{T})\begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix} = \begin{pmatrix} \partial_\rho z_1 \\ \vdots \\ \partial_\rho z_N \end{pmatrix}.$$

The domain of $\mathcal{A}$ is the following

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} \underline{u} \\ \underline{z} \end{pmatrix}, \underline{u} \in \left(\prod_{j=1}^N H^3(0,\ell_j)\right) \cap \mathbb{H}_e^2(\mathcal{T}), \sum_{j=1}^N \frac{\mathrm{d}^2 u_j}{\mathrm{d}x^2}(0) = -\alpha u_1(0), \right.$$
$$\left. \underline{z} \in \prod_{j=1}^N L^2(H^1(0,1)\times\omega_j), z_j(0,x) = u_j\big|_{\omega_j}(x) \right\}.$$

Note that if $\begin{pmatrix} \underline{u} \\ \underline{z} \end{pmatrix} \in D(\mathcal{A})$ then $\underline{u} \in \mathbb{H}_e^2(\mathcal{T})$ that implies $u_j(t,\ell_j) = \partial_x u_j(t,\ell_j) = 0$.

Theorem 2.1 *Assume $\underline{a},\underline{b} \in \mathbb{L}^\infty(\mathcal{T})$ componentwise non-negative that satisfy* (1.1) *and* (1.2). *Let $U_0 \in H$. Then there exists a unique solution $U \in C([0,\infty);H)$ of* (2.2). *Moreover if $U_0 \in D(\mathcal{A})$ then U is a classical solution and*

$$U \in C([0,\infty);D(\mathcal{A})) \cap C^1([0,\infty);H).$$

Proof Let $U = \begin{pmatrix} \underline{u} \\ \underline{z} \end{pmatrix} \in D(\mathcal{A})$, then

$$\langle \mathcal{A}U,U\rangle = \sum_{j=1}^{N}\left(\int_0^{\ell_j}(-\partial_x^3 u_j(x)-\partial_x u_j(x)-a_j(x)u_j(x))u_j(x)dx\right.$$
$$\left.-\int_{\omega_j} b_j(x)z_j(1,x)u_j(x)dx - h_j\int_{\omega_j}\int_0^1 \xi_j(x)\frac{1}{h_j}\partial_\rho z_j(\rho,x)z_j(\rho,x)d\rho dx\right)$$
$$=\sum_{j=1}^{N}\left(\int_0^{\ell_j}\partial_x^2 u_j(x)\partial_x u_j(x)dx - \partial_x^2 u_j(x)u_j(x)\Big|_0^{\ell_j} - \frac{1}{2}u_j^2(x)\Big|_0^{\ell_j} - \int_0^{\ell_j} a_j(x)u_j^2(x)dx\right.$$
$$\left.-\int_{\omega_j} b_j(x)z_j(1,x)u_j(x)dx - \frac{1}{2}\int_{\omega_j}\xi_j(x)z_j^2(\rho,x)\Big|_0^1 dx\right)$$
$$=\sum_{j=1}^{N}\left(\frac{1}{2}(\partial_x u_j(x))^2\Big|_0^{\ell_j} + \partial_x^2 u_j(0)u_1(0) + \frac{1}{2}u_1^2(0) - \int_0^{\ell_j} a_j(x)u_j^2(x)dx\right.$$
$$\left.-\int_{\omega_j} b_j(x)z_j(1,x)u_j(x)dx - \frac{1}{2}\int_{\omega_j}\xi_j(x)z_j^2(1,x)dx + \frac{1}{2}\int_{\omega_j}\xi_j(x)z_j^2(0,x)dx\right).$$

Thus,

$$\langle \mathcal{A}U,U\rangle \le -\frac{1}{2}\sum_{j=1}^{N}(\partial_x u_j(0))^2 + \left(\frac{N}{2}-\alpha\right)u_1^2(0) - \sum_{j=1}^{N}\int_{(0,\ell_j)/\omega_j} a_j(x)u_j^2(x)dx$$
$$+\sum_{j=1}^{N}\int_{\omega_j}\left(-a_j(x)+\frac{b_j(x)}{2}+\frac{\xi_j(x)}{2}\right)u_j^2(x)dx + \sum_{j=1}^{N}\int_{\omega_j}\left(\frac{b_j(x)}{2}-\frac{\xi_j(x)}{2}\right)z_j^2(1,x)dx. \tag{2.3}$$

Using (1.4), (1.5) and that $\alpha \ge \frac{N}{2}$ we conclude that $\langle \mathcal{A}U,U\rangle \le 0$, thus $\mathcal{A}$ is dissipative. Easy calculations show that

$$\mathcal{A}^*\begin{pmatrix}\underline{v}\\ \underline{y}\end{pmatrix} = \begin{pmatrix}(\mathrm{D}_x(\mathcal{T})+\mathrm{D}_x^3(\mathcal{T}))\underline{v} - \underline{a}.*\underline{v} + \underline{\xi}.*\underline{\tilde{y}}(0,\cdot)\\ \dfrac{1}{\underline{h}}.*\mathrm{D}_\rho(\mathcal{T})\underline{y}\end{pmatrix},$$

in which $\tilde{y}_j(0,\cdot)\in L^2(0,\ell_j)$ is the extension by 0 of $y_j(0,\cdot)$ outside ω_j and with

$$D(\mathcal{A}^*) = \left\{\begin{pmatrix}\underline{v}\\ \underline{y}\end{pmatrix}, \underline{v}\in\left(\prod_{j=1}^{N}H^3(0,\ell_j)\right)\cap\mathbb{H}_e^1(\mathcal{T}),\ \sum_{j=1}^{N}\frac{\mathrm{d}^2 v_j}{\mathrm{d}x^2}(0) = (\alpha-N)v_1(0),\right.$$
$$\left.\partial_x v_j(0)=0,\ \forall j=1,\cdots,N,\ \ \underline{y}\in\prod_{j=1}^{N}L^2(H^1(0,1)\times\omega_j), y_j(1,x) = -\frac{b_j(x)}{\xi_j(x)}v_j\big|_{\omega_j}(x)\right\}.$$

Note that $\begin{pmatrix}\underline{v}\\ \underline{y}\end{pmatrix}\in D(\mathcal{A}^*)$ then $\underline{v}\in\mathbb{H}_e^1(\mathcal{T})$ that implies $v_j(t,\ell_j)=0$. Let $V=\begin{pmatrix}\underline{v}\\ \underline{y}\end{pmatrix}\in D(\mathcal{A}^*)$, then

$$
\begin{aligned}
\langle \mathcal{A}^*V,V\rangle &= \sum_{j=1}^{N}\left(\int_0^{\ell_j}(\partial_x^3 v_j(x)+\partial_x v_j(x)-a_j(x)v_j(x))v_j(x)dx\right.\\
&\left.+\int_{\omega_j}\xi_j(x)y_j(0,x)v_j(x)dx+\int_{\omega_j}h_j\int_0^1\xi_j(x)\frac{1}{h_j}\partial_\rho y_j(\rho,x)y_j(\rho,x)d\rho dx\right),\\
&=\sum_{j=1}^{N}\left(-\int_0^{\ell_j}\partial_x^2 v_j(x)\partial_x v_j(x)dx+\partial_x^2 v_j(x)v_j(x)\Big|_0^{\ell_j}+\frac{1}{2}|v_j(x)|^2\Big|_0^{\ell_j}\right.\\
&\left.-\int_0^{\ell_j}a_j(x)v_j^2(x)dx+\int_{\omega_j}\xi_j(x)y_j(0,x)v_j(x)dx+\frac{1}{2}\int_{\omega_j}\xi_j(x)|y_j(\rho,x)|^2\Big|_0^1dx\right)\\
&\leq\sum_{j=1}^{N}\left(-\frac{1}{2}|\partial_x v_j(x)|^2\Big|_0^{\ell_j}-\partial_x^2 v_j(0)v_1(0)-\frac{1}{2}v_1^2(0)-\int_0^{\ell_j}a_j(x)v_j^2(x)dx\right.\\
&\left.+\frac{1}{2}\int_{\omega_j}\xi_j(x)y_j^2(0,x)dx+\frac{1}{2}\int_{\omega_j}\xi_j(x)v_j^2(x)dx+\frac{1}{2}\int_{\omega_j}\xi_j(x)(y_j^2(1,x)-y_j^2(0,x))dx\right),\\
&\leq -\frac{1}{2}\sum_{j=1}^{N}|\partial_x v_j(\ell_j)|^2+\left(\frac{N}{2}-\alpha\right)v_1^2(0)+\int_{\omega_j}\left(-a_j(x)+\frac{\xi_j(x)}{2}+\frac{b_j^2(x)}{2\xi_j(x)}\right)v_j^2(x)dx\\
&\quad-\int_{(0,\ell_j)/\omega_j}a_j(x)v_j^2(x)dx-\frac{1}{2}\int_{\omega_j}\xi_j(x)y_j^2(0,x)dx.
\end{aligned}
$$

Moreover we know that $\xi_j(x)>b_j(x)>b_0>0$, for $x\in\omega_j$, then we have that $\dfrac{b_j^2(x)}{\xi_j(x)}<b_j(x)$, for $x\in\omega_j$ and then

$$
-a_j(x)+\frac{\xi_j(x)}{2}+\frac{b_j^2(x)}{2\xi_j(x)}<-a_j(x)+\frac{\xi_j(x)}{2}+\frac{b_j(x)}{2}\leq 0,\qquad \text{for } x\in\omega_j,
$$

thus as $\alpha\geq\frac{N}{2}$, $\mathcal{A}^*$ is dissipative. Finally $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative, also $\mathcal{A}$ is densely defined closed operator, thus $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup of contractions on H [16]. □

As the systems (LKdVd) and (2.2) are equivalent we obtain the well-posedness of (LKdVd). Let $S(t), t\geq 0$ the semigroup of contractions associated with $\mathcal{A}$. Next result gives us some a priori estimates for (LKdVd).

Proposition 2.1 *Assume $\underline{a}$, $\underline{b}\in\mathbb{L}^\infty(\mathcal{T})$ componentwise non-negative that satisfy*(1.1) *and* (1.2). *Then, the map*

$$
U_0=(\underline{u}^0,\underline{z}^0(-\underline{h}\cdot,\cdot))\mapsto S(\cdot)(\underline{u}^0,\underline{z}^0(-\underline{h}\cdot,\cdot)) \tag{2.4}
$$

is continuous from H to $\mathbb{B}\times C([0,T];\mathbb{L}^2(\Omega))$ and for $(\underline{u}^0,\underline{z}^0(-\underline{h}\cdot,\cdot))\in H$ the following estimates hold

$$\begin{aligned}\sum_{j=1}^{N}\int_0^T\int_0^{\ell_j}a_j(x)(u_j(t,x))^2dxdt+\int_0^T\int_{\omega_j}(z_j(t,1,x))^2dxdt\\ \leq C\left(\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})}+\|\underline{z}^0(-\underline{h}\cdot,\cdot)\|^2_{\mathbb{L}^2(\Omega)}\right),\end{aligned}\tag{2.5}$$

$$\begin{aligned}\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})}\leq\left(\frac{1+2T\|\underline{a}\|_{\mathbb{L}^\infty(\mathcal{T})}+2T\|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})}}{T}\right)\|\underline{u}\|^2_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))}\\ +2\left(\alpha-\frac{N}{2}\right)\|u_1(\cdot,0)\|^2_{L^2(0,T)}+\|\partial_x\underline{u}(\cdot,0)\|^2_{L^2(0,T)}+\|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})}\|\underline{z}^0(-\underline{h}\cdot,\cdot)\|^2_{\mathbb{L}^2(\Omega)},\end{aligned}\tag{2.6}$$

$$\|\underline{z}^0(-\underline{h}\cdot,\cdot)\|^2_{\mathbb{L}^2(\Omega)}\leq\|\underline{z}(T,\cdot,\cdot)\|^2_{\mathbb{L}^2(\Omega)}+\sum_{j=1}^{N}\frac{1}{h_j}\int_0^T\int_{\omega_j}|z_j(t,1,x)|^2dxdt.\tag{2.7}$$

Proof Taking $(\underline{u}^0,\underline{z}^0(-\underline{h}\cdot,\cdot))\in H$, Theorem 2.1 gives us $S(\cdot)(\underline{u}^0,\underline{z}^0(-\underline{h}\cdot,\cdot))=(\underline{u},\underline{z})\in C([0,T];H)$ and using that $\mathcal{A}$ generates a C_0 semigroup of contractions, we get for all $t\in[0,T]$

$$\begin{aligned}\sum_{j=1}^{N}\int_0^{\ell_j}(u_j(t,x))^2dx+\sum_{j=1}^{N}h_j\int_{\omega_j}\int_0^1\xi_j(x)(z_j(t,\rho,x))^2d\rho dx\\ \leq\sum_{j=1}^{N}\int_0^{\ell_j}(u_j^0(x))^2dx+\sum_{j=1}^{N}h_j\int_{\omega_j}\int_0^1\xi_j(x)(z_j^0(-h_j\rho,x))^2d\rho dx.\end{aligned}\tag{2.8}$$

Let $\underline{p}\in\prod_{j=1}^{N}C^\infty([0,T]\times(0,1))$ and $\underline{q}\in\prod_{j=1}^{N}C^\infty([0,T]\times(0,\ell_j))$. Now multiplying (LKdVd) by q_ju_j and (2.1) by p_jz_j and integrating on $(0,s)\times(0,\ell_j)$ and $(0,s)\times(0,1)\times\omega_j$ we can obtain

$$\begin{aligned}\int_0^{\ell_j}q_j(t,x)|u_j(t,x)|^2dx\Big|_0^s-\int_0^s\int_0^{\ell_j}(\partial_tq_j+\partial_xq_j+\partial_x^3q_j)|u_j|^2dxdt\\ +2\int_0^s\int_0^{\ell_j}a_jq_j|u_j|^2dxdt+2\int_0^s\int_0^{\ell_j}b_j(x)q_j(t,x)u_j(t-h_j,x)u_j(t,x)dxdt\\ +3\int_0^s\int_0^{\ell_j}|\partial_xu_j|^2\partial_xq_jdxdt=\int_0^s\left[(q_j+\partial_x^2q_j)|u_j|^2+2q_ju_j\partial_x^2u_j\right.\\ \left.-2\partial_xq_ju_j\partial_xu_j-q_j|\partial_xu_j|^2\right](t,0)dt,\end{aligned}\tag{2.9}$$

$$\begin{aligned}\int_0^1\int_{\omega_j}(z_j(t,\rho,x))^2p_j(t,\rho)dxd\rho\Big|_0^s-\frac{1}{h_j}\int_0^s\int_0^1\int_{\omega_j}(h_j\partial_tp_j+\partial_\rho p_j)|z_j|^2dxd\rho dt\\ +\frac{1}{h_j}\int_0^s\int_{\omega_j}(z_j(t,1,x))^2p_j(t,1)-(u_j(t,x))^2p_j(t,0)dxdt=0.\end{aligned}\tag{2.10}$$

Taking $s=T$ and $p_j=\rho$ in (2.10) we get

$$\begin{aligned}\int_0^1\int_{\omega_j}\rho\left[z_j(T,\rho,x)^2-z_j^0(-\rho h_j,x)^2\right]dxd\rho-\frac{1}{h_j}\int_0^T\int_0^1\int_{\omega_j}|z_j|^2dxd\rho dt\\+\frac{1}{h_j}\int_0^T\int_{\omega_j}z_j(t,1,x)^2dxd\rho=0.\end{aligned}\tag{2.11}$$

Thus,

$$\begin{aligned}&\frac{1}{h_j}\int_0^T\int_{\omega_j}z_j(t,1,x)^2dxd\rho\\&\leq\frac{1}{h_j}\int_0^T\int_0^1\int_{\omega_j}|z_j|^2dxd\rho dt+\int_0^1\int_{\omega_j}\rho z_j^0(-\rho h_j,x)^2dxd\rho.\end{aligned}$$

and hence with (2.8) we get

$$\sum_{j=1}^N\int_{\omega_j}z_j(t,1,x)^2dxd\rho\leq C\left(\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})}+\|\underline{z}^0(-\underline{h}\cdot,\cdot)\|^2_{\mathbb{L}^2(\Omega)}\right).\tag{2.12}$$

Then taking $q_j=1$ in (2.9)

$$\begin{aligned}&\sum_{j=1}^N\int_0^{\ell_j}|u_j(s,x)|^2dx+\int_0^s\sum_{j=1}^N|\partial_xu_j(t,0)|^2dt+(2\alpha-N)\int_0^s|u_1(t,0)|^2dt\\&+2\sum_{j=1}^N\int_0^s\int_0^{\ell_j}a_j|u_j|^2dxdt+\sum_{j=1}^N2\int_0^s\int_0^{\ell_j}b_ju_j(t-h_j,x)u_j(t,x)dxdt\\&=\sum_{j=1}^N\int_0^{\ell_j}|u_j(0,x)|^2dx.\end{aligned}$$

Thus

$$\begin{aligned}&\sum_{j=1}^N\int_0^{\ell_j}|u_j(s,x)|^2dx+\int_0^s\sum_{j=1}^N|\partial_xu_j(t,0)|^2dt+\sum_{j=1}^N2\int_0^T\int_0^{\ell_j}a_j|u_j|^2dxdt\\&\leq\sum_{j=1}^N\int_0^{\ell_j}|u_j(0,x)|^2dx+\sum_{j=1}^N2\int_0^T\int_0^{\ell_j}b_j|u_j(t-h_j,x)||u_j(t,x)|dxdt.\end{aligned}$$

Note now that

$$\begin{aligned}&2\int_0^s\int_0^{\ell_j}b_j(x)|u_j(t-h_j,x)||u_j(t,x)|dxdt\\&\leq\int_0^s\int_0^{\ell_j}b_j(x)|u_j(t-h_j,x)|^2dxdt+\int_0^s\int_0^{\ell_j}b_j(x)|u_j(t,x)|^2dxdt,\\&=\int_0^s\int_0^{\ell_j}b_j(x)|u_j(t,x)|^2dxdt+\int_{-h_j}^{s-h_j}\int_{\omega_j}b_j(x)|u_j(t,x)|^2dxdt,\\&\leq 2\int_0^s\int_0^{\ell_j}b_j(x)|u_j(t,x)|^2dxdt+\int_{-h_j}^0\int_{\omega_j}b_j(x)|z_j^0(t,x)|^2dxdt,\end{aligned}$$

which implies

$$2\sum_{j=1}^{N}\int_0^s\int_0^{\ell_j} b_j(x)|u_j(t-h_j,x)||u_j(t,x)|dxdt \le C\left(\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})}+\|\underline{z}^0(-\underline{h}\cdot,\cdot)\|^2_{\mathbb{L}^2(\Omega)}\right). \tag{2.13}$$

Thus, we have

$$\sum_{j=1}^{N}\int_0^{\ell_j}|u_j(s,x)|^2dx+\int_0^T\sum_{j=1}^{N}|\partial_x u_j(t,0)|^2dt+\sum_{j=1}^{N}\int_0^T\int_0^{\ell_j}a_j(x)(u_j(t,x))^2dxdt \le C\left(\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})}+\|\underline{z}^0(-\underline{h}\cdot,\cdot)\|^2_{\mathbb{L}^2(\Omega)}\right), \tag{2.14}$$

that brings (2.5) using (2.12).
Note also that $\sum_{j=1}^{N}\partial_x u_j(\cdot,0)\in L^2(0,T)$. Moreover integrating (2.14) with respect to s over $[0,T]$ we can obtain.

$$\|\underline{u}\|^2_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))}\le CT\left(\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})}+\|\underline{z}^0(-\underline{h}\cdot,\cdot)\|^2_{\mathbb{L}^2(\Omega)}\right). \tag{2.15}$$

We are going to consider the following multiplier presented in [4], $q_j(t,x)=\frac{x(2\ell_j-x)}{\ell_j^2}$, this multiplier satisfies the next properties

- $q_j(t,0)=0,\ \forall t\in[0,T]$.
- $0\le q_j(t,x)\le 1,\ \forall (t,x)\in[0,T]\times[0,\ell_j]$.
- $0\le \partial_x q_j(t,x)\le \frac{2}{\ell_j},\ \forall (t,x)\in[0,T]\times[0,\ell_j]$.
- $\partial_x^2 q_j(t,x)=-\frac{2}{\ell_j^2},\ \forall (t,x)\in[0,T]\times[0,\ell_j]$.

Taking $q_j(t,x)=\frac{x(2\ell_j-x)}{\ell_j^2}$ and $s=T$ in (2.9) we get

$$\begin{aligned}&\sum_{j=1}^{N}\int_0\ell_j q_j(t,x)|u_j(T,x)|^2dx+2\sum_{j=1}^{N}\int_0^T\int_{\omega_j}q_j(t,x)b_j(x)u_j(t-h_j,x)u_j(t,x)dxdt\\&+2\int_0^T u_1(t,0)\sum_{j=1}^{N}\frac{2}{\ell_j}\partial_x u_j(t,0)dt+2\sum_{j=1}^{N}\int_0^T\int_0^{\ell_j}q_j(t,x)a_j(x)|u_j(t,x)|^2dxdt\\&-\sum_{j=1}^{N}\int_0^T\int_0^{\ell_j}\partial_x q_j(t,x)|u_j(t,x)|^2dxdt+3\sum_{j=1}^{N}\int_0^T\int_0^{\ell_j}|\partial_x u_j(t,x)|^2\partial_x q_j(t,x)dxdt\\&=\sum_{j=1}^{N}\int_0\ell_j q_j(0,x)|u_j^0|^2dx-\left(\sum_{j=1}^{N}\frac{2}{\ell_j^2}\right)\int_0^T|u_1(t,0)|^2dt.\end{aligned}$$

and then recalling that $L=\max_{j=1,\cdots,N}\ell_j$ and taking $\ell=\min_{j=1,\cdots,N}\ell_j$

$$\frac{2}{L^2}\|u_1(\cdot,0)\|^2_{L^2(0,T)} \leq \frac{2}{\ell^2}\|\underline{u}\|^2_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} - 2\int_0^T u_1(t,0)\sum_{j=1}^N \partial_x u_j(t,0)\frac{2}{\ell_j}dt$$
$$-2\sum_{j=1}^N\int_0^T\int_{\omega_j} q_j(t,x)b_j(x)u_j(t-h_j,x)u_j(t,x)dxdt + \|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})}. \tag{2.16}$$

Using Young's inequality, (2.13) and (2.15) we get that $u_1(\cdot,0)\in L^2(0,T)$ and

$$\|u_1(\cdot,0)\|^2_{L^2(0,T)} \leq C\left(\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} + \|\underline{z}^0(-\underline{h}\cdot,\cdot)\|^2_{\mathbb{L}^2(\Omega)}\right).$$

Now, let us choose $q_j = x$ and $s=T$ in (2.9)

$$\int_0^{\ell_j} x|u_j|^2dx\Big|_0^T dx - \int_0^T\int_0^{\ell_j}|u_j|^2dxdt + 2\int_0^T\int_0^{\ell_j} xb_j(x)u_j(t-h_j,x)u_j(t,x)dxdt$$
$$+2\int_0^T\int_0^{\ell_j}a_j(x)x|u_j|^2dxdt + 3\int_0^T\int_0^{\ell_j}|\partial_x u_j|^2dxdt = \int_0^T -2u_j(t,0)\partial_x u_j(t,0)dt$$

Then

$$3\sum_{j=1}^N\int_0^T\int_0^{\ell_j}|\partial_x u_j|^2dxdt \leq \left(1+2L\|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})}\right)\sum_{j=1}^N\int_0^T\int_0^{\ell_j}|u_j|^2dxdt$$
$$+L\|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})}\sum_{j=1}^N\int_{-h_j}^0\int_{\omega_j}|z_j^0(t,x)|^2dxdt + L\sum_{j=1}^N\int_0^{\ell_j}|u_j(0,x)|^2dx$$
$$-2\sum_{j=1}^N\int_0^T u_1(t,0)\partial_x u_j(t,0)dt$$

and hence

$$3\sum_{j=1}^N\int_0^T\int_0^{\ell_j}|\partial_x u_j|^2dxdt \leq C\left(\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} + \|\underline{z}^0(-\underline{h}\cdot,\cdot)\|^2_{\mathbb{L}^2(\Omega)}\right),$$

that brings with (2.8) the continuity of the map (2.4) from H to $\mathbb{B}\times C([0,T]:\mathbb{L}^2(\Omega))$. Now taking $q_j = T-t$ and $s=T$ in (2.9), we obtain,

$$-\int_0^{\ell_j} T|u_j(0,x)|^2dx + \int_0^T\int_0^{\ell_j}|u_j|^2dxdt + 2\int_0^T\int_0^{\ell_j}a_j(x)(T-t)|u_j|^2dxdt$$
$$+2\int_0^T\int_0^{\ell_j}b_j(x)(T-t)u_j(t-h_j,x)u_j(t,x)dxdt = \int_0^T\left[(T-t)|u_j(t,0)|^2\right.$$
$$\left.+2(T-t)u_j(t,0)\partial_x^2u_j(t,0) - (T-t)|\partial_x u_j(t,0)|^2\right]dt,$$

then

$$T\sum_{j=1}^{N}\int_0^{\ell_j}|u_j(0,x)|^2dx=\sum_{j=1}^{N}\left(\int_0^T\int_0^{\ell_j}|u_j|^2dxdt++2\int_0^T\int_0^{\ell_j}(T-t)a_j|u_j|^2dxdt\right.$$
$$\left.+2\int_0^T\int_0^{\ell_j}b_j(x)(T-t)u_j(t-h_j,x)u_j(t,x)dxdt\right)+(2\alpha-N)\int_0^T(T-t)|u_1(t,0)|^2dt$$
$$+\sum_{j=1}^{N}\int_0^T(T-t)|\partial_x u_j(t,0)|^2dt.$$

Finally we get (2.6), that is

$$\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})}\leq\left(\frac{1+2T\|\underline{a}\|_{\mathbb{L}^\infty(\mathcal{T})}+2T\|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})}}{T}\right)\|\underline{u}\|^2_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))}$$
$$+2\left(\alpha-\frac{N}{2}\right)\|u_1(\cdot,0)\|^2_{L^2(0,T)}+\|\partial_x\underline{u}(\cdot,0)\|^2_{L^2(0,T)}+\|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})}\|\underline{z}^0(-\underline{h}\cdot,\cdot)\|^2_{\mathbb{L}^2(\Omega)}.$$

Lastly taking $p_j=1$ and $s=T$ in (2.10)

$$\int_0^1\int_{\omega_j}|z_j(T,\rho,x)|^2dxd\rho-\int_0^1\int_{\omega_j}|z_j^0(-h_j\rho,x)|^2dxd\rho$$
$$+\frac{1}{h_j}\int_0^T\int_{\omega_j}\left[|z_j(t,1,x)|^2-|u_j(t,x)|^2\right]dxdt=0$$

and hence we obtain (2.7). □

2.2 Extra boundary conditions

Following [1] we need now some regularity results for the linear delayed KdV equation with extra boundary source term $g(t)$ at the central node

$$\begin{cases}\partial_t u_j(t,x)+\partial_x u_j(t,x)+\partial_x^3 u_j(t,x)+a_j(x)u_j(t,x) & \\ +b_j(x)u_j(t-h_j,x)=0, & x\in(0,\ell_j),\ t>0,\ j=1,\cdots,N,\\ u_j(t,0)=u_k(t,0), & \forall j,k=1,\cdots N,\\ \sum_{j=1}^N\partial_x^2u_j(t,0)=-\alpha u_1(t,0)+g(t), & t>0,\\ u_j(t,\ell_j)=\partial_x u_j(t,\ell_j)=0, & t>0,\ j=1,\cdots,N,\\ u_j(0,x)=u_j^0(x), & x\in(0,\ell_j),\\ u_j(t,x)=z_j^0(t,x), & (t,x)\in(-h_j,0)\times(0,\ell_j).\end{cases}\tag{2.17}$$

Recall that $z_j(t,\rho,x)=u_j\big|_{\omega_j}(t-h_j\rho,x)$, for $x\in\omega_j$, $\rho\in(0,1)$ is solution of

$$\begin{cases}h_j\partial_t z_j(t,\rho,x)+\partial_\rho z_j(t,\rho,x)=0, & x\in\omega_j,\rho\in(0,1),t>0,\\ z_j(t,0,x)=u_j(t,x), & x\in\omega_j,t>0,\\ z_j(0,\rho,x)=u_j\big|_{\omega_j}(-h_j\rho,x)=z_j^0(-h_j\rho,x), & \rho\in(0,1).\end{cases}\tag{2.18}$$

Define $\mathfrak{h}=\max_{j=1,\cdots N}h_j$.

Proposition 2.2 *Assume* $\underline{a}, \underline{b} \in \mathbb{L}^\infty(\mathcal{T})$ *componentwise non-negative that satisfy* (1.1) *and* (1.2). *Let* $(U_0, g) \in D(\mathcal{A}) \times C_0^2([0,T])$ *where* $C_0^2([0,T]) := \{\varphi \in C^2([0,T]) : \varphi(0) = 0\}$. *Then there exists a unique classical solution* $U = \begin{pmatrix} \underline{u} \\ \underline{z} \end{pmatrix} \in C([0,T], D(\mathcal{A})) \cap C^1([0,T]; H)$ *of* (2.17)-(2.18).

Proof Let $\underline{v} = \underline{u} - g\underline{\phi}$, where $\underline{\phi}$ is defined as

$$\phi_j(x) = \frac{(x-\ell_j)^2}{\ell_j^2\left(2\sum_{j=1}^N \ell_j^{-2} + \alpha\right)}.$$

We can easily check that

$$\begin{cases} \phi_j(\ell_j) = \phi_j'(\ell_j) = 0, & \forall j = 1, \cdots, N \\ \phi_j(0) = \dfrac{1}{2\sum\limits_{j=1}^N \ell_j^{-2} + \alpha} = \phi_k(0), \ \forall j,k = 1, \cdots N, & \\ \sum_{j=1}^N \phi_j''(0) = 1 - \alpha\phi_1(0), & t > 0. \end{cases} \tag{2.19}$$

We extend g on $[-\mathfrak{h}, 0]$ by $g(t) \equiv 0$ for $t \in [-\mathfrak{h}, 0]$. Then $\underline{v}$ satisfies

$$\begin{cases} \partial_t v_j(t,x) + \partial_x v_j(t,x) + \partial_x^3 v_j(t,x) + a_j(x) v_j(t,x) & \\ +b_j(x) v_j(t-h_j, x) = f_j(t,x), & x \in (0,\ell_j),\ t > 0,\ j = 1, \cdots, N, \\ v_j(t,0) = v_k(t,0), & \forall j,k = 1, \cdots N, \\ \sum_{j=1}^N \partial_x^2 v_j(t,0) = -\alpha v_1(t,0), & t > 0, \\ v_j(t,\ell_j) = \partial_x v_j(t,\ell_j) = 0, & t > 0,\ j = 1, \cdots, N, \\ v_j(0,x) = u_j^0(x), & x \in (0,\ell_j), \\ v_j(t,x) = z_j^0(t,x), & (t,x) \in (-h_j, 0) \times (0,\ell_j). \end{cases} \tag{2.20}$$

for $f_j(t,x) = -\phi_j(x) g'(t) - (\phi_j' + \phi_j''' + a_j\phi_j)(x) g(t)$. Then, taking $y_j(t,\rho,x) = v_j\big|_{\omega_j}(t - h_j\rho, x)$

$$\begin{cases} h_j \partial_t y_j(t,\rho,x) + \partial_\rho y_j(t,\rho,x) = 0, & x \in \omega_j, \rho \in (0,1), t > 0, \\ y_j(t,0,x) = v_j(t,x), & x \in \omega_j, t > 0, \\ y_j(0,\rho,x) = v_j\big|_{\omega_j}(-h_j\rho, x) = z_j^0(-h_j\rho, x), \ \rho \in (0,1). & \end{cases} \tag{2.21}$$

Thus defining $V = \begin{pmatrix} \underline{v} \\ \underline{y} \end{pmatrix}$, as $-\underline{\phi} g' - (\underline{\phi}' + \underline{\phi}''' + \underline{a}. * \underline{\phi}) g \in C^1([0,T], \mathbb{L}^2(\mathcal{T}))$, by the classical semigroup theory and the well-posedness of the linear case, we deduce the existence of a unique solution V of (2.20)-(2.21). Moreover $V \in C([0,T], D(\mathcal{A})) \cap C^1([0,T]; H)$ and hence (2.17)-(2.18) admits a unique solution $U \in C([0,T], D(\mathcal{A})) \cap C^1([0,T]; H)$. □

Now, we study the same system but with less regular data.

Proposition 2.3 *Assume* $\underline{a}$, $\underline{b} \in \mathbb{L}^\infty(\mathcal{T})$ *componentwise non-negative that satisfy* (1.1) *and* (1.2). *Let* $(U_0, g) \in H \times L^2(0,T)$, *then there exists a unique mild solution* $U \in \mathbb{B} \times C\left([0,T];\mathbb{L}^2(\Omega)\right)$ *of* (2.17)-(2.18). *Furthermore* $u_1(\cdot,0)$ *and* $\partial_x \underline{u}(\cdot,0)$ *belong to* $L^2(0,T)$ *and we have the following estimates*

$$\|\underline{u}\|_{\mathbb{B}}^2 \leq C\left(\|g\|_{L^2(0,T)}^2 + \|\underline{u}^0\|_{\mathbb{L}(\mathcal{T})}^2 + \|\underline{z}^0(-\underline{h}\cdot,\cdot)\|_{\mathbb{L}^2(\Omega)}^2\right). \tag{2.22}$$

$$\|\underline{z}\|_{C([0,T],\mathbb{L}^2(\Omega))}^2 \leq C\left(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + \|\underline{z}^0(-\underline{h}\cdot,\cdot)\|_{\mathbb{L}^2(\Omega)}^2 + \|g\|_{L^2(0,T)}^2\right). \tag{2.23}$$

$$\begin{aligned}\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 \leq &\left(\frac{1+2T\|\underline{a}\|_{\mathbb{L}^\infty(\mathcal{T})}+2T\|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})}}{T}\right)\|\underline{u}\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))}^2 \\ &+\|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})}\|\underline{z}^0(-\underline{h}\cdot,\cdot)\|_{\mathbb{L}^2(\Omega)}^2 + C\left(\|u_1(\cdot,0)\|_{L^2(0,T)}^2 + \|g\|_{L^2(0,T)}^2\right)\end{aligned} \tag{2.24}$$

$$\|\underline{z}^0(-\underline{h}\cdot,\cdot)\|_{\mathbb{L}^2(\Omega)}^2 \leq \|\underline{z}(T,\cdot,\cdot)\|_{\mathbb{L}^2(\Omega)}^2 + \sum_{j=1}^N \frac{1}{h_j}\int_0^T\int_{\omega_j}|z_j(t,1,x)|^2 dxdt. \tag{2.25}$$

Proof Those estimates are obtained in a similar way as Proposition 2.1 and for that many calculations are omitted. First suppose that $(U_0, g) \in D(\mathcal{A}) \times C_0^2([0,T])$ and thus the solution of (2.17)-(2.18) satisfies $U \in C([0,T];D(\mathcal{A}) \cap C^1([0,T];H)$.

Multiplying (2.17) by u_j and integrating on $[0,s]\times[0,\ell_j]$ gives us

$$\begin{aligned}&\sum_{j=1}^N\int_0^{\ell_j}|u_j(s,x)|^2dx + \int_0^s\sum_{j=1}^N|\partial_x u_j(t,0)|^2dt + (2\alpha-N)\int_0^s|u_1(t,0)|^2dt \\ &+2\sum_{j=1}^N\int_0^s\int_0^{\ell_j}a_j|u_j|^2dxdt + 2\sum_{j=1}^N\int_0^s\int_0^{\ell_j}b_ju_j(t-h_j,x)u_j(t,x)dxdt \\ &\qquad= \sum_{j=1}^N\int_0^{\ell_j}|u_j(0,x)|^2dx + 2\int_0^s u_1(t,0)g(t)dt,\end{aligned}$$

then using that

$$\begin{aligned}&2\int_0^s\int_0^{\ell_j}b_j(x)u_j(t-h_j,x)u_j(t,x)dxdt \\ &\leq 2\int_0^s\int_0^{\ell_j}b_j(x)|u_j(t,x)|^2dxdt + \int_{-h_j}^0\int_{\omega_j}b_j(x)|z_j^0(t,x)|^2dxdt,\end{aligned}$$

$$\begin{aligned}&\sum_{j=1}^N\int_0^{\ell_j}|u_j(s,x)|^2dx + \int_0^s\sum_{j=1}^N|\partial_x u_j(t,0)|^2dt + (2\alpha-N)\int_0^s|u_1(t,0)|^2dt + \\ &\qquad 2\sum_{j=1}^N\int_0^s\int_0^{\ell_j}(a_j-b_j)|u_j|^2dxdt \leq 2\int_0^s u_1(t,0)g(t)dt \\ &\qquad\qquad +C\left(\|\underline{u}^0\|_{\mathbb{L}(\mathcal{T})}^2 + \|\underline{z}^0(-\underline{h}\cdot,\cdot)\|_{\mathbb{L}^2(\Omega)}^2\right).\end{aligned} \tag{2.26}$$

Note now that (2.16) still holds in this case and we can obtain

$$\|u_1(\cdot,0)\|^2_{L^2(0,T)} \leq C\left(\|\underline{u}\|^2_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} + \|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} + \sum_{j=1}^N \int_0^T |\partial_x u_j(t,0)|^2 dt \right.$$
$$\left. + \|\underline{z}^0(-\underline{h}\cdot,\cdot)\|^2_{\mathbb{L}^2(\Omega)}\right).$$

From (2.26) we obtain

$$\|u_1(\cdot,0)\|^2_{L^2(0,T)} \leq C\left(\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} + \|\underline{z}^0(-\underline{h}\cdot,\cdot)\|^2_{\mathbb{L}^2(\Omega)} + \int_0^T u_1(t,0)g(t)dt\right)$$

and again by Young's inequality

$$\|u_1(\cdot,0)\|^2_{L^2(0,T)} \leq C\left(\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} + \|\underline{z}^0(-\underline{h}\cdot,\cdot)\|^2_{\mathbb{L}^2(\Omega)} + \|g\|^2_{L^2(0,T)}\right)$$

Thus $u_1(\cdot,0) \in L^2(0,T)$ and from (2.26) $\underline{u}(s,\cdot) \in \mathbb{L}^2(\mathcal{T})$ for $s \in [0,T]$, $\partial_x \underline{u}(\cdot,0) \in L^2(0,T)$ and

$$\max_{s\in[0,T]} \|\underline{u}(s,\cdot)\|^2_{\mathbb{L}^2(\mathcal{T})} \leq C\left(\|g\|^2_{L^2(0,T)} + \|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} + \|\underline{z}^0(-\cdot\,\underline{h}\cdot)\|^2_{\mathbb{L}^2(\Omega)}\right). \tag{2.27}$$

Now multiplying (2.17) by xu_j yields

$$\int_0^{\ell_j} x|u_j|^2 dx\Big|_0^T\, dx - \int_0^T \int_0^{\ell_j} |u_j|^2 dxdt + 2\int_0^T \int_0^{\ell_j} x b_j(x) u_j(t-h_j,x)u_j(t,x)dxdt$$
$$+2\int_0^T \int_0^{\ell_j} a_j(x) x|u_j|^2 dxdt + 3\int_0^T \int_0^{\ell_j} |\partial_x u_j|^2 dxdt = \int_0^T -2u_j(t,0)\partial_x u_j(t,0)dt.$$

Then

$$3\sum_{j=1}^N \int_0^T \int_0^{\ell_j} |\partial_x u_j|^2 dxdt \leq \left(1+2L\|\underline{b}\|_{L^\infty(\mathcal{T})}\right)\sum_{j=1}^N \int_0^T \int_0^{\ell_j} |u_j|^2 dxdt$$
$$+L\sum_{j=1}^N \int_0^{\ell_j} |u_j(0,x)|^2 dx + L\|\underline{b}\|_{L^\infty(\mathcal{T})}\sum_{j=1}^N \int_{-h_j}^0 \int_{\omega_j} |z_j^0(t,x)|^2 dxdt$$
$$+\|u_1(\cdot,0)\|^2_{L^2(0,T)} + \|\partial_x \underline{u}(\cdot,0)\|^2_{L^2(0,T)}$$

and using (2.27) we deduce (2.22). Now multiplying (2.17) by $(T-t)u_j$ yields

$$-\int_0^{\ell_j} T|u_j(0,x)|^2 dx + \int_0^T \int_0^{\ell_j} |u_j|^2 dxdt + 2\int_0^T \int_0^{\ell_j} a_j(x)(T-t)|u_j|^2 dxdt$$
$$+2\int_0^T \int_0^{\ell_j} b_j(x)(T-t)u_j(t-h_j,x)u_j(t,x)dxdt = \int_0^T \left[(T-t)|u_j(t,0)|^2\right.$$
$$\left. +2(T-t)u_j(t,0)\partial_x^2 u_j(t,0) - (T-t)|\partial_x u_j(t,0)|^2\right] dt,$$

then

$$T\sum_{j=1}^{N}\int_0^{\ell_j}|u_j(0,x)|^2dx=\sum_{j=1}^{N}\left(\int_0^T\int_0^{\ell_j}|u_j|^2dxdt+2\int_0^T\int_0^{\ell_j}(T-t)a_j|u_j|^2dxdt+\right.$$
$$\left.2\int_0^T\int_0^{\ell_j}b_j(x)(T-t)u_j(t-h_j,x)u_j(t,x)dxdt\right)+(2\alpha-N)\int_0^T(T-t)|u_1(t,0)|^2dt$$
$$+\sum_{j=1}^{N}\int_0^T(T-t)|\partial_x u_j(t,0)|^2dt-2\int_0^T(T-t)u_1(t,0)g(t)dt.$$

Finally we get

$$\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})}\leq\left(\frac{1+2T\|\underline{a}\|_{\mathbb{L}^\infty(\mathcal{T})}+2T\|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})}}{T}\right)\|\underline{u}\|^2_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))}$$
$$+\|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})}\|\underline{z}^0(-\underline{h}\cdot,\cdot)\|^2_{\mathbb{L}^2(\Omega)}+C\left(\|u_1(\cdot,0)\|^2_{L^2(0,T)}+\|g\|^2_{L^2(0,T)}\right)$$

and hence (2.24). We can conclude that the estimates for (2.18) are the same as Proposition 2.1. By density of $D(\mathcal{A})$ in H, $C_0^2([0,T])$ in $L^2(0,T)$, we extend our result to arbitrary data $(U_0,g)\in H\times L^2(0,T)$. □

2.3 Extra source term

We add now a source term $f_j(t,x)$ on each edge in our KdV problem.

$$\begin{cases}\partial_t u_j(t,x)+\partial_x u_j(t,x)+\partial_x^3 u_j(t,x)+a_j(x)u_j(t,x) & \\ +b_j(x)u_j(t-h_j,x)=f_j(t,x), & x\in(0,\ell_j),\ t>0,\ j=1,\cdots,N,\\ u_j(t,0)=u_k(t,0), & \forall j,k=1,\cdots N,\\ \sum_{j=1}^N\partial_x^2u_j(t,0)=-\alpha u_1(t,0)+g(t), & t>0,\\ u_j(t,\ell_j)=\partial_x u_j(t,\ell_j)=0, & t>0,\ j=1,\cdots,N,\\ u_j(0,x)=u_j^0(x), & x\in(0,\ell_j)\\ u_j(t,x)=z_j^0(t,x), & (t,x)\in(-h_j,0)\times(0,\ell_j).\end{cases}\tag{2.28}$$

We set as in the previous cases $z_j(t,\rho,x)=u_j\big|_{\omega_j}(t-h_j\rho,x)$ $x\in\omega_j$, $\rho\in(0,1)$. Then

$$\begin{cases}h_j\partial_t z_j(t,\rho,x)+\partial_\rho z_j(t,\rho,x)=0, & x\in\omega_j,\rho\in(0,1),t>0,\\ z_j(t,0,x)=u_j(t,x), & x\in\omega_j,t>0,\\ z_j(0,\rho,x)=u_j\big|_{\omega_j}(-h_j\rho,x)=z_j^0(-h_j\rho,x), & \rho\in(0,1).\end{cases}\tag{2.29}$$

Proposition 2.4 *Assume $\underline{a}$, $\underline{b}\in\mathbb{L}^\infty(\mathcal{T})$ componentwise non-negative that satisfy* (1.1) *and* (1.2). *Let $(U_0,g,\underline{f})\in H\times L^2(0,T)\times L^1(0,T;\mathbb{L}^2(\mathcal{T}))$ then there exists a unique mild solution $U=\begin{pmatrix}\underline{u}\\ \underline{z}\end{pmatrix}\in\mathbb{B}\times C\left([0,T];\mathbb{L}^2(\Omega)\right)$ to* (2.28)-(2.29). *Furthermore we have the following estimates,*

$$\|(\underline{u},\underline{z})\|^2_{C([0,T],H)} \leq C\left(\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} + \|\underline{z}^0(-\underline{h}\cdot,\cdot)\|^2_{\mathbb{L}^2(\Omega)} + \|\underline{f}\|^2_{L^1(0,T;\mathbb{L}^2(\mathcal{T}))} + \|g\|^2_{L^2(0,T)}\right), \tag{2.30}$$

$$\begin{aligned}\|\partial_x \underline{u}\|^2_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} \leq C(1+T)&\left(\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} + \|\underline{z}^0(-\underline{h}\cdot,\cdot)\|^2_{\mathbb{L}^2(\Omega)}\right.\\ &\left.+\|\underline{f}\|^2_{L^1(0,T;\mathbb{L}^2(\mathcal{T}))} + \|g\|^2_{L^2(0,T)}\right).\end{aligned} \tag{2.31}$$

Proof The well-posedness of (2.28)-(2.29) follows from classical semigroup theory and from the propositions given considering the source term $\begin{pmatrix} f \\ \underline{0}\end{pmatrix}$. Also this gives us the first inequality, for the second one note that multiplying (2.28) by u_j and integrating we get

$$\begin{aligned}&\sum_{j=1}^N \int_0^{\ell_j} |u_j(T,x)|^2 dx + (2\alpha - N)\int_0^T |u_1(t,0)|^2 dt + \sum_{j=1}^N \int_0^T |\partial_x u_j(t,0)|^2 dt\\ &+2\sum_{j=1}^N \int_0^T \int_0^{\ell_j} a_j(x)|u_j(t,x)|^2 dx dt + 2\sum_{j=1}^N \int_0^T \int_0^{\ell_j} b_j u_j(t,x) u_j(t-h_j,x) dx dt\\ &-2\int_0^T u_1(t,0) g(t) dt = 2\sum_{j=1}^N \int_0^T \int_0^{\ell_j} f_j(t,x) u_j(t,x) dx dt + \|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})}.\end{aligned}$$

Note that

$$\begin{aligned}&2\sum_{j=1}^N \int_0^T \int_0^{\ell_j} f_j(t,x) u_j(t,x) dx dt \leq 2\sum_{j=1}^N \int_0^T \|f_j\|_{L^2(0,\ell_j)} \|u_j\|_{L^2(0,\ell_j)} dt,\\ &\leq 2\sum_{j=1}^N \|u_j\|_{C([0,T],L^2(0,\ell_j))} \int_0^T \|f_j\|_{L^2(0,\ell_j)} dt \leq \|\underline{u}\|^2_{C([0,T],\mathbb{L}^2(\mathcal{T}))} + \|\underline{f}\|^2_{L^1(0,T;\mathbb{L}^2(\mathcal{T}))}.\end{aligned}$$

Following the same steps as in Proposition 2.1 and Proposition 2.3 we can get

$$\begin{aligned}&\sum_{j=1}^N \int_0^{\ell_j} |u_j(T,x)|^2 dx + \sum_{j=1}^N \int_0^T |\partial_x u_j(t,0)|^2 dt \leq C\left(\|\underline{u}\|^2_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} + \|(\underline{u}^0, \underline{z}^0(-\underline{h}\cdot,\cdot))\|^2_H\right.\\ &\left.+2\int_0^T u_1(t,0) g(t) dt + \|\underline{u}\|^2_{C([0,T],\mathbb{L}^2(\mathcal{T}))} + \|\underline{f}\|^2_{L^1(0,T;\mathbb{L}^2(\mathcal{T}))}\right).\end{aligned}$$

Now multiplying (2.28) by $q_j u_j$ for $q_j = \frac{x(2\ell_j - x)}{\ell_j^2}$ and using the last inequality we get

$$\|u_1(\cdot,0)\|^2_{L^2(0,T)} \leq C\left(\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} + \|\underline{z}^0(-\underline{h}\cdot,\cdot)\|^2_{\mathbb{L}^2(\Omega)} + \|g\|^2_{L^2(0,T)} + \|\underline{f}\|^2_{L^1(0,T;\mathbb{L}^2(\mathcal{T}))}\right)$$

and we can also have

$$\begin{aligned}\|u_1(\cdot,0)\|^2_{L^2(0,T)} + \|\partial_x u(\cdot,0)\|^2_{L^2(0,T)} \leq C&\left(\|\underline{f}\|^2_{L^1(0,T;\mathbb{L}^2(\mathcal{T}))} + \|g\|^2_{L^2(0,T)}\right.\\ &\left.+\|(\underline{u}^0, \underline{z}^0(-\underline{h}\cdot,\cdot))\|^2_H\right).\end{aligned}$$

Now multiplying (2.28) by xu_j gives us

$$3\sum_{j=1}^{N}\int_0^T\int_0^{\ell_j}|\partial_x u_j|^2dxdt+\sum_{j=1}^{N}\int_0^{\ell_j}x|u_j(T,x)|^2dx-\sum_{j=1}^{N}\int_0^T\int_0^{\ell_j}|u_j|^2dxdt$$
$$+2\sum_{j=1}^{N}x|u_j|^2a_jdxdt+2\sum_{j=1}^{N}\int_0^T\int_0^{\ell_j}xb_ju_j(t-h_j,x)u_j(t,x)dxdt$$
$$=2\sum_{j=1}^{N}\int_0^T\int_0^{\ell_j}xu_jf_jdxdt-2\sum_{j=1}^{N}\int_0^T u_1(t,0)\partial_x u_j(t,0)dt.$$

Hence

$$3\|\partial_x\underline{u}\|^2_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))}\leq T\|\underline{u}\|^2_{C([0,T];\mathbb{L}^2(\mathcal{T}))}+L\left(\|\underline{u}\|^2_{C([0,T],\mathbb{L}^2(\mathcal{T}))}+\|\underline{f}\|^2_{L^1(0,T;\mathbb{L}^2(\mathcal{T}))}\right)$$
$$+N\|u_1(\cdot,0)\|^2_{L^2(0,T)}+\|\partial_x\underline{u}(\cdot,0)\|^2_{L^2(0,T)}.$$

□

2.4 Well-posedness of nonlinear system

The aim of this section is to use the estimates obtained in the last sections to pass to the nonlinear system. The following propositions are needed in order to deal with the internal nonlinearity and boundary nonlinearity respectively.

Proposition 2.5 (Proposition 4.1, [18]) *Let $T,\ L>0$, and $y\in L^2(0,T;H^1(0,L))$. Then $yy_x\in L^1(0,T;L^2(0,L))$ and the map*

$$y\in L^2(0,T;H^1(0,L))\mapsto yy_x\in L^1(0,T;L^2(0,L))$$

is continuous. Moreover we have

$$\|yy_x\|_{L^1(0,T;L^2(0,L))}\leq C\|y\|^2_{L^2(0,T;H^1(0,L))}. \tag{2.32}$$

Proposition 2.6 (Proposition 2.6, [1]) *Let $\underline{u}\in\mathbb{B}$, then $|u_1(t,0)|^2\in L^2(0,T)$ and the map*

$$\underline{u}\in\mathbb{B}\mapsto|u_1(t,0)|^2\in L^2(0,T)$$

is continuous. Moreover, we have the estimate,

$$\|u_1^2(\cdot,0)\|_{L^2(0,T)}\leq\frac{1}{\sqrt{2}}\|\underline{u}\|^2_{\mathbb{B}}. \tag{2.33}$$

Now we are ready to establish our well-posedness result of the nonlinear (KdVd) for small initial data.

Theorem 2.2 *Assume $\underline{a},\ \underline{b}\in\mathbb{L}^\infty(\mathcal{T})$ componentwise non-negative that satisfy* (1.1) *and* (1.2). *Let $(\ell_j)_{j=1}^N\subset(0,+\infty)$, $T>0$, there exists $\epsilon>0$ and $C>0$ such that for all $U_0=(\underline{u}^0,\underline{z}^0(-\underline{h}\cdot,\cdot))\in H$ with $\|U_0\|_H\leq\epsilon$, the nonlinear equation* (KdVd) *has a unique mild solution $\underline{u}\in\mathbb{B}$. Moreover it satisfy*

$$\|\underline{u}\|_{\mathbb{B}}\leq C\|(\underline{u}^0,\underline{z}^0(-\underline{h}\cdot,\cdot))\|_H.$$

Proof Let $U_0 \in H$, with $\|U_0\|_H < \epsilon$, where $\epsilon > 0$ will be chosen later, $\underline{u} \in \mathbb{B}$ and consider the map $\Phi : \mathbb{B} \to \mathbb{B}$ defined by $\Phi(\underline{u}) = \underline{v}$ where $\underline{v}$ is solution of

$$\begin{cases} \partial_t v_j(t,x) + \partial_x v_j(t,x) + \partial_x^3 v_j(t,x) + a_j(x) v_j(t,x) & \\ +b_j(x) v_j(t-h_j,x) = -u_j(t,x)\partial_x u_j(t,x), & x \in (0,\ell_j),\ t>0,\ j=1,\cdots,N, \\ v_j(t,0) = v_k(t,0), & \forall j,k = 1,\cdots N, \\ \sum_{j=1}^N \partial_x^2 v_j(t,0) = -\alpha v_1(t,0) - \dfrac{N}{3}(u_1(t,0))^2, & t>0, \\ v_j(t,\ell_j) = \partial_x v_j(t,\ell_j) = 0, & t>0,\ j=1,\cdots,N, \\ v_j(0,x) = u_j^0(x), & x \in (0,\ell_j), \\ v_j(t,x) = z_j^0(t,x), & (t,x) \in (-h_j,0)\times(0,\ell_j). \end{cases} \tag{2.34}$$

Clearly $\underline{u} \in \mathbb{B}$ is solution of (KdVd) if $\underline{u}$ is a fixed point Φ. From Proposition 2.5 and Proposition 2.6, we get for all $\underline{u} \in \mathbb{B}$

$$\|\Phi(\underline{u})\|_{\mathbb{B}} = \|\underline{v}\|_{\mathbb{B}} \le C\left(\|U_0\|_H + \|\underline{u}\|_{\mathbb{B}}^2\right)$$

and for $\underline{u}$, $\underline{\tilde{u}} \in \mathbb{B}$

$$\|\Phi(\underline{u}) - \Phi(\underline{\tilde{u}})\|_{\mathbb{B}} \le C\left(\|\underline{u}\|_{\mathbb{B}} + \|\underline{\tilde{u}}\|_{\mathbb{B}}\right)\|\underline{u} - \underline{\tilde{u}}\|_{\mathbb{B}}$$

Let us choose $R > 0$ to be defined later and consider Φ restricted to the closed ball $B_{\mathbb{B}}(0,R)$. Then, for any $\underline{u}$, $\underline{\tilde{u}} \in B_{\mathbb{B}}(0,R)$, we have

$$\|\Phi(\underline{u})\|_{\mathbb{B}} \le C(\epsilon + R^2)$$

$$\|\Phi(\underline{u}) - \Phi(\underline{\tilde{u}})\|_{\mathbb{B}} \le 2CR\|\underline{u} - \underline{\tilde{u}}\|_{\mathbb{B}}.$$

Thus if $R < \frac{1}{2C}$ and $\epsilon > 0$ such that $C(\epsilon + R^2) < R$ we obtain the local well-posedness result applying the Banach fixed point Theorem. □

Remark 2.1 On a similar way as Theorem 2.1, we can obtain classical solutions by taking $U_0 \in D(\mathcal{A})$.

3 Stabilization of delayed KdV system

3.1 Lyapunov stabilization of the delayed system

The aim of this part is to prove Theorem 1.1. As we said before this proof is developed in a constructive manner by using a Lyapunov function.

Proof of Theorem 1.1 :
Let $\underline{u}$ a regular enough solution of (KdVd) with $U_0 \in D(\mathcal{A})$ satisfying $\|U_0\|_H \leq \epsilon$, where $\epsilon > 0$ will be chosen later. Following [2, 21] we consider the next Lyapunov candidate for (KdVd)

$$V(t) = E(t) + \mu_1 V_1(t) + \mu_2 V_2(t). \tag{3.1}$$

where E is defined by (1.6)

$$V_1(t) = \sum_{j=1}^{N} \int_0^{\ell_j} x|u_j(t,x)|^2 dx, \text{ and } V_2(t) = \sum_{j=1}^{N} h_j \int_{\omega_j} \int_0^1 (1-\rho)|u_j(t-h_j\rho,x)|^2 d\rho dx.$$

Clearly

$$E(t) \leq V(t) \leq \left(1 + \max\left\{L\mu_1, \frac{\mu_2}{b_0}\right\}\right) E(t).$$

After some computations we have,

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leq -(2\alpha - N)|u_1(t,0)|^2 - \sum_{j=1}^{N} |\partial_x u_j(t,0)|^2 - \sum_{j=1}^{N} \int_{(0,\ell_j)/\omega_j} a_j(x)|u_j(t,x)|^2 dx$$
$$+\sum_{j=1}^{N} \int_{\omega_j} (-2a_j(x) + b_j(x) + \xi_j(x))|u_j(t,x)|^2 dx + \sum_{j=1}^{N} \int_{\omega_j} (b_j(x) - \xi_j(x))|u_j(t-h_j,x)|^2 dx,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}V_1(t) = \sum_{j=1}^{N} \int_0^{\ell_j} |u_j(t,x)|^2 dx - 3\sum_{j=1}^{N} \int_0^{\ell_j} |\partial_x u_j(t,x)|^2 dx - 2u_1(t,0)\sum_{j=1}^{N} \partial_x u_j(t,0)$$
$$+\frac{2}{3}\sum_{j=1}^{N} \int_0^{\ell_j} u_j^3(t,x)dx - 2\sum_{j=1}^{N} \int_0^{\ell_j} xa_j(x)|u_j(t,x)|^2 dx - 2\sum_{j=1}^{N} \int_{\omega_j} xb_j(x)u_j(t,x)u_j(t-h_j,x)dx$$
$$\leq \sum_{j=1}^{N} \int_0^{\ell_j} |u_j(t,x)|^2 dx - 3\sum_{j=1}^{N} \int_0^{\ell_j} |\partial_x u_j(t,x)|^2 dx + \frac{N}{2}|u_1(t,0)|^2 + \frac{1}{2}\sum_{j=1}^{N} |\partial_x u_j(t,0)|^2$$
$$+\frac{2}{3}\sum_{j=1}^{N} \int_0^{\ell_j} u_j^3(t,x)dx + L\sum_{j=1}^{n} \int_{\omega_j} b_j(x)|u_j(t,x)|^2 dx + +L\sum_{j=1}^{n} \int_{\omega_j} b_j(x)|u_j(t-h_j,x)|^2 dx,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}V_2(t) = \sum_{j=1}^{N} \int_{\omega_j} |u_j(t,x)|^2 dx - \sum_{j=1}^{N} \int_{\omega_j} \int_0^1 |u_j(t-h_j\rho,x)|^2 d\rho dx.$$

Our idea now is to prove that for suitable choice of μ_1, μ_2, $\gamma > 0$ we have that $\frac{\mathrm{d}}{\mathrm{d}t}V(t) + 2\gamma V(t) \leq 0$, which gives the exponential stability.
Using the following Poincaré's inequality: If $y \in H^1(0,L)$ and $y(0) = 0$ or $y(L) = 0$,

we have $\|y\|_{L^2(0,L)} \leq \frac{2L}{\pi}\|\partial_x y\|_{L^2(0,L)}$. We can check easily that for $\gamma > 0$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V(t)+2\gamma V(t) \leq & -2\left(\alpha-\frac{N}{2}-\mu_1\frac{N}{2}\right)|u_1(t,0)|^2+(\mu_1-1)\sum_{j=1}^{N}|\partial_x u_j(t,0)|^2 \\
&+\sum_{j=1}^{N}\int_{\omega_j}(-2a_j+b_j+\xi_j+L\mu_1 b_j+\mu_2)|u_j|^2dx+\frac{2}{3}\mu_1\sum_{j=1}^{N}\int_0^{\ell_j}(u_j)^3dx \\
&+\sum_{j=1}^{N}\int_{\omega_j}(b_j-\xi_j+\mu_1 Lb_j)|u_j(t-h_j,x)|^2dx \\
&+\left[\frac{4L^2(\mu_1+2\mu_1\gamma L+2\gamma)}{\pi^2}-3\mu_1\right]\sum_{j=1}^{N}\int_0^{\ell_j}|\partial_x u_j(t,x)|^2dx \\
&+\sum_{j=1}^{N}\int_{\omega_j}\int_0^1(2\gamma\mu_2 h_j+2\gamma h_j\xi_j-\mu_2)|u_j(t-h_j\rho,x)|^2d\rho dx.
\end{aligned}$$

For the term involving $\int_0^{\ell_j}u_j^3(t,x)dx$, note that

$$\int_0^{\ell_j}u_j^3(t,x)dx \leq \|u_j\|^2_{L^\infty(0,\ell_j)}\int_0^{\ell_j}|u_j(t,x)|dx \leq \|u_j\|^2_{L^\infty(0,\ell_j)}\|u_j\|_{L^2(0,\ell_j)}\sqrt{\ell_j}.$$

By the injection of $H^1(0,\ell_j)$ into $L^\infty(0,\ell_j)$ we know that $\|u_j\|_{L^\infty(0,\ell_j)} \leq \sqrt{\ell_j}\|\partial_x u_j\|_{L^2(0,\ell_j)}$, then

$$\int_0^{\ell_j}u_j^3(t,x)dx \leq \|u_j\|^2_{L^\infty(0,\ell_j)}\|u_j\|_{L^2(0,\ell_j)}\sqrt{\ell_j} \leq \ell_j\|\partial_x u_j\|^2_{L^2(0,\ell_j)}\sqrt{\ell_j}\|u_j\|_{L^2(0,\ell_j)}.$$

Recalling that $L=\max_{j=1,\cdots,N}\ell_j$ and as the energy is not increasing we get $\|u_j\|_{L^2(0,\ell_j)} \leq \|U_0\|_H$. Choosing $\|U_0\|_H \leq \epsilon$ we get

$$\frac{2}{3}\mu_1\sum_{j=1}^{N}\int_0^{\ell_j}u_j^3(t,x)dx \leq \frac{2}{3}\mu_1\epsilon L^{3/2}\sum_{j=1}^{N}\int_0^{\ell_j}|\partial_x u_j(t,x)|^2dx.$$

Now taking

$$0<\mu_1<\min_{j=1,\cdots,N}\inf_{\omega_j}\left\{1,\frac{2a_j-b_j-\xi_j}{Lb_j},\frac{\xi_j-b_j}{Lb_j},\frac{1}{N}(2\alpha-N)\right\},$$

$$0<\mu_2<\min_{j=1,\cdots,N}\inf_{\omega_j}\left\{2a_j-b_j-\xi_j-\mu_1 Lb_j\right\}.$$

Then $-(2\alpha-N-\mu_1 N)<0$ and $(\mu_1-1)<0$. Moreover for all $j=1,\cdots,N$

$$(-2a_j+b_j+\xi_j+L\mu_1 b_j+\mu_2)<0, \quad (b_j-\xi_j+\mu_1 Lb_j)<0$$

Finally joining the estimates

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V(t)+2\gamma V(t) \leq &\left[\frac{4L^2(\mu_1+2\mu_1\gamma L+2\gamma)}{\pi^2}-3\mu_1+\frac{2L^{3/2}\epsilon\mu_1}{3}\right]\|\partial_x\underline{u}(t,x)\|^2_{\mathbb{L}^2(\mathcal{T})}\\
&+\sum_{j=1}^{N}\int_{\omega_j}\int_0^1(2h_j\gamma(\mu_2+\xi_j)-\mu_2)|u_j(t-h_j\rho,x)|^2d\rho dx
\end{aligned}
$$

and then as $L<\frac{\sqrt{3}}{2}\pi$, we can choose $\epsilon<\frac{3}{2}\frac{(3\pi^2-4L^2)}{\pi^2L^{3/2}}$ and then take $\gamma>0$ satisfying (1.7) to obtain $\frac{\mathrm{d}}{\mathrm{d}t}V(t)+2\gamma V(t)\leq 0$. We get the desired exponential stability, by density we can extend the result to any $U_0\in H$, with $\|U_0\|_H\leq\epsilon$. □

Remark 3.1 As in [2, 21] we obtain an estimation of the rate of decay. Also recall that we can improve the result searching for a better Poincaré's inequality, and as is commented in [2, 21] looking for a new multiplier for the Lyapunov function V_1, in the sense that the restriction on the lengths, comes from the multiplier x.

Remark 3.2 Note that in absence of the feedback terms (with and without delay) this result can be see as an alternative proof via Lyapunov theory of Theorem 3.4 [1] (in our case with a more restrictive condition on the lengths).

3.2 Observability approach

In the previous section we obtained a stabilization result under the hypothesis that $L<\frac{\sqrt{3}}{2}\pi$, $\alpha>N/2$ and for small initial data. Now we are going to prove a result without restrictions on the lengths that holds for $\alpha\geq N/2$ and small initial data. The idea is to obtain an observability inequality in the linear system. The proof is based on a contradiction argument and hence we can not estimate the decay rate of the energy, contrary to Theorem 1.1.

Theorem 3.1 *Assume $\underline{a}$, $\underline{b}\in\mathbb{L}^\infty(\mathcal{T})$ componentwise non-negative that satisfy* (1.1) *and* (1.2). *Let $(\ell_j)_{j=1}^N\subset(0,\infty)$ and $T>\mathfrak{h}$, then there exists $C>0$ such that for all $U_0=(\underline{u}^0,\underline{z}^0(-\underline{h}\cdot,\cdot))\in H$ we have the following observability inequality*

$$
\begin{aligned}
&\sum_{j=1}^{N}\int_0^{\ell_j}(u_j^0(x))^2dx+\sum_{j=1}^{N}h_j\int_{\omega_j}\int_0^1\xi_j(x)(z_j^0(-h_j\rho,x))^2dxd\rho\\
&\leq C\Bigg(\sum_{j=1}^{N}\int_0^T(\partial_xu_j(t,0))^2dt+\sum_{j=1}^{N}\int_0^T\int_0^{\ell_j}a_j(x)(u_j(t,x))^2dxdt\\
&+\sum_{j=1}^{N}\int_0^T\int_{\omega_j}(z_j(t,1,x))^2dxdt+(2\alpha-N)\int_0^T(u_1(t,0))^2dt\Bigg),
\end{aligned}
\tag{Obs}
$$

for $\begin{pmatrix}\underline{u}\\ \underline{z}\end{pmatrix}=S(\cdot)U_0$, *solution of* (LKdVd).

Proof We follow the classical approach presented in [18].

Suppose that (Obs) is false. Then we can find a sequence $(U_0^n)_{n\in\mathbb{N}} = (\underline{u}^{0,n}, \underline{z}^{0,n}(-\underline{h}\cdot,\cdot))_{n\in\mathbb{N}} \subset H$ such that

$$\sum_{j=1}^{N}\int_0^{\ell_j}(u_j^{0,n}(x))^2dx + \sum_{j=1}^{N}h_j\int_{\omega_j}\int_0^1\xi_j(x)(z_j^{0,n}(-h_j\rho,x))^2dxd\rho = 1$$

and for $(\underline{u}^n, \underline{z}^n) = S(\underline{u}^{0,n}, \underline{z}^{0,n}(-\underline{h}\cdot,\cdot))$ we have

$$\begin{aligned}&\sum_{j=1}^{N}\int_0^T(\partial_x u_j^n(t,0))^2dt + \sum_{j=1}^{N}\int_0^T\int_0^{\ell_j}a_j(x)(u_j^n(t,x))^2dxdt\\&+\sum_{j=1}^{N}\int_0^T\int_{\omega_j}(z_j^n(t,1,z))^2dxdt + (2\alpha - N)\int_0^T(u_1^n(t,0))^2 \to 0\end{aligned} \tag{3.2}$$

when $n\to\infty$. Now using (2.22) for $g=0$ we get that $(\underline{u}^n)_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;\mathbb{H}_e^1(\mathcal{T}))$ and then as $\partial_t u_j^n = -\partial_x u_j^n - \partial_x^3 u_j^n - a_j u_j^n - b_j z_j^n(1)$, we have that $(\partial_t u_j^n)_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;H^{-2}(0,\ell_j))$. Using the Aubin-Lions Lemma, we can deduce that $(\underline{u}^n)_{n\in\mathbb{N}}$ is relatively compact in $L^2(0,T;\mathbb{L}^2(\mathcal{T}))$ and hence we can assume that it is convergent in $L^2(0,T;\mathbb{L}^2(\mathcal{T}))$.

Moreover for $T > \mathfrak{h}$ since $z_j^n(t,\rho,x) = u_j^n\big|_{\omega_j}(t-h_j\rho,x)$ we have

$$\begin{aligned}\int_{\omega_j}\int_0^1(z_j^n(T,\rho,x))^2d\rho dx &= \int_{\omega_j}\int_0^1(u_j^n(T-\rho h_j,x))^2d\rho dx\\&\le\frac{1}{h_j}\int_{\omega_j}\int_0^T(u_j(t,x))^2dtdx.\end{aligned}$$

Now thanks to (2.7)

$$\begin{aligned}\|\underline{z^{0,n}}(-\underline{h}\cdot,\cdot)\|^2_{\mathbb{L}^2(\Omega)} &\le \|\underline{z}^n(T,\cdot,\cdot)\|^2_{\mathbb{L}^2(\Omega)} + \sum_{j=1}^{N}\frac{1}{h_j}\int_0^T\int_{\omega_j}|z_j^n(t,1,x)|^2dxdt.\\&\le\sum_{j=1}^{N}\frac{1}{h_j}\int_0^T\int_{\omega_j}(u_j^n(t,x))^2dxdt + \sum_{j=1}^{N}\frac{1}{h_j}\int_0^T\int_{\omega_j}|z_j^n(t,1,x)|^2dxdt,\end{aligned}$$

and hence $(\underline{z}^{0,n}(-\underline{h}\cdot,\cdot))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{L}^2(\Omega)$ using (3.2). Moreover using (2.6) and (3.2) we get that $(\underline{u}^{0,n})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{L}^2(\mathcal{T})$.

Let $U_0 = (\underline{u}^0, \underline{z}^0(-\underline{h}\cdot,\cdot)) = \lim_{n\to\infty}(\underline{u}^{0,n}, \underline{z}^{0,n}(-\underline{h}\cdot,\cdot))$ in H and $(\underline{u},\underline{z}) = S(\cdot)(\underline{u}^0, \underline{z}^0(-\underline{h}\cdot,\cdot))$. By Proposition 2.1 we have:

$$\sum_{j=1}^{N}\int_0^T\int_0^{\ell_j} a_j(x)(u_j^n(t,x))^2dxdt+\sum_{j=1}^{N}\int_0^T\int_{\omega_j} b_j(x)(z_j^n(t,1,x))^2dxdt$$
$$\underset{n\to\infty}{\longrightarrow}\sum_{j=1}^{N}\int_0^T\int_0^{\ell_j} a_j(x)(u_j(t,x))^2dxdt+\sum_{j=1}^{N}\int_0^T\int_{\omega_j} b_j(x)(z_j(t,1,x))^2dxdt.$$

Thus

$$\sum_{j=1}^{N}\int_0^{\ell_j}(u_j^0(x))^2dx+\sum_{j=1}^{N}h_j\int_{\omega_j}\int_0^1\xi_j(x)(z_j^0(-h_j\rho,x))^2dxd\rho=1$$

and

$$\sum_{j=1}^{N}\int_0^T\int_0^{\ell_j} a_j(x)(u_j(t,x))^2dxdt+\sum_{j=1}^{N}\int_0^T\int_{\omega_j} b_j(x)(z_j(t,1,x))^2dxdt=0.$$

As $z_j(t,1,x)=u_j(t-h_j,x)=0$ in $(0,T)\times\omega_j$, we can deduce that $\underline{z}^0=0$ and $\underline{z}=0$. Moreover $u_j=0$ on $(0,T)\times\omega_j$, and as u_j is solution of $\partial_t u_j+\partial_x u_j+\partial_x^3 u_j=0$ thanks to Holmgrem's Theorem, $u_j=0$ in $(0,T)\times(0,\ell_j)$. Thus $(\underline{u},\underline{z})=(\underline{0},\underline{0})$ and we get a contradiction which ends the proof. □

Remark 3.3 Note that in the case $\alpha=N/2$ the term of $\|u_1(t,0)\|^2_{L^2(0,T)}$ disappear of (Obs).

Now from the observability inequality (Obs), we can obtain the exponential stability of the linear system (LKdVd).

Theorem 3.2 *Assume $\underline{a}$, $\underline{b}\in\mathbb{L}^\infty(\mathcal{T})$ componentwise non-negative that satisfy* (1.1) *and* (1.2). *Let $(\ell_j)_{j=1}^N\subset(0,\infty)$, then for all $(\underline{u}^0,\underline{z}^0(-\underline{h}\cdot,\cdot))\in H$, the energy of the system* (LKdVd) *defined by* (1.6) *decays exponentially, i.e, there exists $C>0$ and $\mu>0$ such that $E(t)\le CE(0)e^{-\mu t}$ for all $t>0$.*

Proof We follow [21, 10] (see also [17]). Note that for $U_0\in D(\mathcal{A})$ the energy of (LKdVd) satisfies for $C_1>0$

$$\begin{aligned}\frac{d}{dt}E(t)\le -C_1\Bigg((2\alpha-N)\int_0^T|u_1(t,0)|^2dt+\sum_{j=1}^{N}\int_0^T|\partial_x u_j(t,0)|^2\\ +\sum_{j=1}^{N}\int_0^{\ell_j}a_j(x)|u_j(t,x)|^2dx+\sum_{j=1}^{N}\int_{\omega_j}|u_j(t-h_j,x)|^2dx\Bigg).\end{aligned}\tag{3.3}$$

Integrating between 0 and $T>\mathfrak{h}$ we have

$$E(T)-E(0)\leq -C_1\left((2\alpha-N)\int_0^T |u_1(t,0)|^2dt+\sum_{j=1}^N\int_0^T |\partial_x u_j(t,0)|^2dt\right.$$
$$\left.+\sum_{j=1}^N\int_0^T\int_0^{\ell_j} a_j(x)|u_j(t,x)|^2dxdt+\sum_{j=1}^N\int_{\omega_j}\int_0^T |u_j(t-h_j,x)|^2dxdt\right).$$

The last expression can be rewritten as

$$(2\alpha-N)\int_0^T |u_1(t,0)|^2dt+\sum_{j=1}^N\int_0^T\int_0^{\ell_j} a_j(x)|u_j(t,x)|^2dxdt$$
$$+\sum_{j=1}^N\int_0^T |\partial_x u_j(t,0)|^2dt+\sum_{j=1}^N\int_{\omega_j}\int_0^T |u_j(t-h_j,x)|^2dxdt\leq \frac{1}{C_1}(E(0)-E(T)).$$

Using that the energy is non-increasing and (Obs) we get

$$E(T)\leq E(0)\leq C\left((2\alpha-N)\int_0^T |u_1(t,0)|^2dt+\sum_{j=1}^N\int_0^T\int_0^{\ell_j} a_j(x)|u_j(t,x)|^2dxdt\right.$$
$$\left.+\sum_{j=1}^N\int_0^T |\partial_x u_j(t,0)|^2dt+\sum_{j=1}^N\int_{\omega_j}\int_0^T |u_j(t-h_j,x)|^2dxdt\right)\leq \frac{C}{C_1}(E(0)-E(T))$$

which implies

$$E(T)\leq \gamma E(0),\quad \text{with } \gamma=\frac{\frac{C}{C_1}}{1+\frac{C}{C_1}}<1. \tag{3.4}$$

Now as the system is invariant in time, we can repeat this argument on $[(m-1)T,mT]$ for $m=1,2,\cdots$ to obtain

$$E(mT)\leq \gamma E((m-1)T)\leq\cdots\leq \gamma^m E(0).$$

Hence we have $E(mT)\leq e^{-\mu mT}E(0)$ where $\mu=\frac{1}{T}\ln(\frac{1}{\gamma})>0$. Let $t>\mathfrak{h}$. Then there exists $m\in\mathbb{N}^*$ such that $(m-1)T<t\leq mT$, and then using again the non-increasing property of the energy we get

$$E(t)\leq E((m-1)T)\leq e^{-\mu(m-1)T}E(0)\leq \frac{1}{\gamma}e^{-\mu t}E(0).$$

By density of $D(\mathcal{A})$ in H we can extend our result to any initial data in H. □

Remark 3.4 Note that if $\alpha=N/2$ the term of $\|u_1(t,0)\|^2_{L^2(0,T)}$ disappears of (3.3) which is consistent with Remark 3.3.

To end this part we give the proof of Theorem 1.2, inspired by [21, 3, 9].

Proof of Theorem 1.2 :
Let $(\underline{u}^0, \underline{z}^0(-\underline{h}\cdot,\cdot)) \in H$ with $\|(\underline{u}^0, \underline{z}^0(-\underline{h}\cdot,\cdot))\|_H \le r$ for some $r > 0$ that will be chosen after, then the solution $\underline{u}$ of (KdVd) can be decomposed into $\underline{u} = \underline{\bar{u}} + \underline{\tilde{u}}$ respectively solutions of

$$\begin{cases} \partial_t \bar{u}_j(t,x) + \partial_x \bar{u}_j(t,x) + \partial_x^3 \bar{u}_j(t,x) + a_j(x)\bar{u}_j(t,x) & \\ +b_j(x)\bar{u}_j(t-h_j,x) = 0, & x \in (0,\ell_j),\ t>0,\ j=1,\cdots,N, \\ \bar{u}_j(t,0) = \bar{u}_k(t,0), & \forall j,k = 1,\cdots N, \\ \sum_{j=1}^N \partial_x^2 \bar{u}_j(t,0) = -\alpha \bar{u}_1(t,0), & t>0, \\ \bar{u}_j(t,\ell_j) = \partial_x \bar{u}_j(t,\ell_j) = 0, & t>0,\ j=1,\cdots,N, \\ \bar{u}_j(0,x) = u_j^0(x), & x \in (0,\ell_j), \\ \bar{u}_j(t,x) = z_j^0(t,x), & (t,x) \in (-h_j,0)\times(0,\ell_j). \end{cases} \tag{3.5}$$

$$\begin{cases} \partial_t \tilde{u}_j(t,x) + \partial_x \tilde{u}_j(t,x) + \partial_x^3 \tilde{u}_j(t,x) + a_j(x)\tilde{u}_j(t,x) & \\ +b_j(x)\tilde{u}_j(t-h_j,x) = -u_j\partial_x u_j, & x \in (0,\ell_j),\ t>0,\ j=1,\cdots,N, \\ \tilde{u}_j(t,0) = \tilde{u}_k(t,0), & \forall j,k = 1,\cdots N, \\ \sum_{j=1}^N \partial_x^2 \tilde{u}_j(t,0) = -\alpha \tilde{u}_1(t,0) - \dfrac{N}{3}u_1^2(t,0), & t>0, \\ \tilde{u}_j(t,\ell_j) = \partial_x \tilde{u}_j(t,\ell_j) = 0, & t>0,\ j=1,\cdots,N, \\ \tilde{u}_j(0,x) = 0, & x \in (0,\ell_j), \\ \tilde{u}_j(t,x) = 0, & (t,x) \in (-h_j,0)\times(0,\ell_j). \end{cases} \tag{3.6}$$

In simple words $\underline{\bar{u}}$ is solution of (LKdVd) with initial data $(\underline{u}^0, \underline{z}^0(-\underline{h}\cdot,\cdot))$ and $\underline{\tilde{u}}$ is solution of (2.28) with null initial data and source terms $f_j = u_j\partial_x u_j$ and $g = -\frac{N}{3}u_1^2(t,0)$. Then using Proposition 2.5, Proposition 2.6 and Theorem 3.2, we have

$$\begin{aligned} \|\underline{u}(T), \underline{z}(T)\|_H &\le \|\underline{\tilde{u}}(T), \underline{\tilde{z}}(T)\|_H + \|\underline{\bar{u}}(T), \underline{\bar{z}}(T)\|_H \\ \le C\left(\|\underline{u}\partial_x \underline{u}\|_{L^1(0,T;\mathbb{L}^2(\mathcal{T}))} + \|u_1^2(t,0)\|_{L^2(0,T)}\right) &+ \gamma\|U_0\|_H \le \gamma\|U_0\|_H + C\|\underline{u}\|_{\mathbb{B}}^2, \end{aligned}$$

where $\gamma < 1$. Our plan now is to deal with the term $\|\underline{u}\|_{\mathbb{B}}^2$. Multiplying (KdVd) by u_j and integrating in $(0,s)\times(0,\ell_j)$ we can get

$$\begin{aligned} &\|\underline{u}(s,\cdot)\|_{\mathbb{L}^2(\mathcal{T})}^2 + \sum_{j=1}^N \int_0^s |\partial_x u_j(t,0)|^2 ds + (2\alpha - N)\int_0^s |u_1(t,0)|^2 ds \\ &+2\sum_{j=1}^N \int_0^s \int_0^{\ell_j} (a_j - b_j)|u_j|^2 dx ds \le \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + \|\underline{z}^0(-\underline{h}\cdot,\cdot)\|_{\mathbb{L}^2(\Omega)}^2. \end{aligned} \tag{3.7}$$

Now multiplying (KdVd) by $q_j u_j$ with $q_j = \frac{x(2\ell_j - x)}{\ell_j^2}$ we can obtain

$$\|u_1(t,0)\|_{L^2(0,T)}^2 \le C\|U_0\|_H^2 + \frac{2}{3}\sum_{j=1}^N \int_0^T \int_0^{\ell_j} u_j^3(t,x)dxdt.$$

As $\forall j = 1,\cdots,N$ $u_j \in L^2(0,T;H^1(0,\ell_j))$ and $H^1(0,\ell_j)$ embeds into $C([0,\ell_j])$ we get following [3, 17]

$$\sum_{j=1}^{N}\int_0^T\int_0^{\ell_j}|u_j|^3 dxdt \leq CT^{1/2}\|U_0\|_H^2\|\underline{u}\|_{L^2(0,T;\mathbb{H}_e^1(\mathcal{T}))}$$

and then

$$\|u_1(t,0)\|_{L^2(0,T)}^2 \leq C\|U_0\|_H^2 + CT^{1/2}\|U_0\|_H^2\|\underline{u}\|_{L^2(0,T;\mathbb{H}_e^1(\mathcal{T}))}.$$

On a similar way multiplying (KdVd) by xu_j and using the last inequality we deduce

$$\|\partial_x\underline{u}\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))}^2 \leq C\left(\|U_0\|_H^2 + \|U_0\|_H^2\|\underline{u}\|_{L^2(0,T;\mathbb{H}_e^1(\mathcal{T}))}\right).$$

Using Young's inequality, we can find $C > 0$ such that

$$\|\partial_x\underline{u}\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))}^2 \leq C\left(\|U_0\|_H^2 + \|U_0\|_H^4\right). \tag{3.8}$$

Combining the estimates (3.7) and (3.8) we get

$$\|\underline{u}(T),\underline{z}(T)\|_H \leq \|U_0\|_H\left(\gamma + C\|U_0\|_H + C\|U_0\|_H^3\right). \tag{3.9}$$

Taking $\|U_0\|_H \leq \epsilon$ for ϵ small enough such that $\gamma + C\epsilon + C\epsilon^3 < 1$, then the proof follows in the same way as Theorem 3.2. □

3.3 Semi-global Stabilization

The aim of this section is to prove Theorem 1.3, that is a semi-global result without restriction on the lengths and for $\alpha \geq N/2$. The main idea is to obtain an observability inequality as (Obs) working directly with the nonlinear system (KdVd). In this context two main difficulties appears, the first one is to pass to the limit in the nonlinear term and the second that Holmgrem's Theorem does not apply in the nonlinear case.

Proof of Theorem 1.3 :
To prove this result we adapt the techniques of[3]. We need the next Unique Continuation Property of Saut and Scheurer.

Theorem 3.3 (Theorem 4.2, [20]) *Let $L > 0$ and $y \in L^2(0,T;H^3(0,L))$ be a solution of*

$$y_t + y_x + y_{xxx} + yy_x = 0,$$

such that $y(t,x) = 0$, for $(t,x) \in (t_1,t_2)\times\omega$, where ω is a nonempty open subset of $(0,L)$. Then $y(t,x) = 0$, for $(t,x) \in (t_1,t_2)\times(0,L)$.

First defining $z_j(t,\rho,x) = u_j\big|_{\omega_j}(t-h_j\rho,x)$ $x \in \omega_j$, $\rho \in (0,1)$, for $\underline{u}$ solution of (KdVd), we can check that

$$\begin{cases} h_j\partial_t z_j(t,\rho,x) + \partial_\rho z_j(t,\rho,x) = 0, & x \in \omega_j, \rho \in (0,1), t > 0, \\ z_j(t,0,x) = u_j(t,x), & x \in \omega_j, t > 0, \\ z_j(0,\rho,x) = u_j\big|_{\omega_j}(-h_j\rho,x) = z_j^0(-h_j\rho,x), \ \rho \in (0,1). \end{cases} \tag{3.10}$$

Multiplying (3.10) by z_j and integrating on $(0,s)\times(0,1)\times\omega_j$ we can obtain

$$\|\underline{z}^0(-\underline{h}\cdot,\cdot)\|^2_{\mathbb{L}^2(\Omega)} \leq \|\underline{z}(s,\cdot,\cdot)\|^2_{\mathbb{L}^2(\Omega)} + \sum_{j=1}^N \frac{1}{h_j}\int_0^T\int_{\omega_j}|z_j(t,1,x)|^2dxdt.$$

Now integrating this relation on $(0,T)$

$$\begin{aligned} T\|\underline{z}^0(-\underline{h}\cdot,\cdot)\|^2_{\mathbb{L}^2(\Omega)} \leq & \sum_{j=1}^N \int_0^T\int_0^1\int_{\omega_j}|z_j(t,\rho,x)|^2dxd\rho dt \\ & + \sum_{j=1}^N \frac{T}{h_j}\int_0^T\int_{\omega_j}|z_j(t,1,x)|^2dxdt. \end{aligned}$$

Note now that for $T > \mathfrak{h}$ we have

$$\begin{aligned} \int_0^T\int_0^1\int_{\omega_j}|z_j(t,\rho,x)|^2dxd\rho dt &= \int_0^T\int_0^1\int_{\omega_j}|u_j(t-\rho h_j,x)|^2dxd\rho dt \\ &= \int_0^T\int_{t-h_j}^t\int_{\omega_j}|u_j(s,x)|^2dxdsdt \leq \frac{T}{h_j}\int_{-h_j}^T\int_{\omega_j}|u_j(s,x)|^2dxds \\ &= \frac{T-h_j}{h_j}\int_{-h_j}^T\int_{\omega_j}|u_j(s,x)|^2dxds + \frac{T}{h_j}\int_{T-h_j}^T\int_{\omega_j}|u_j(s,x)|^2dxds \\ &\leq \frac{T}{h_j}\int_0^T\int_{\omega_j}(|u_j(t,x)|^2 + |u_j(t-h_j,x)|^2)dxdt \\ &\leq C\left(\int_0^T\int_{\omega_j}a_j|u_j|^2dxdt + \int_0^T\int_{\omega_j}|z_j(t,1,x)|^2dxdt\right), \end{aligned}$$

which gives us

$$T\|\underline{z}^0(-\underline{h}\cdot,\cdot)\|^2_{\mathbb{L}^2(\Omega)} \leq C\left(\sum_{j=1}^N\int_0^T\int_{\omega_j}a_j|u_j|^2dxdt + \int_0^T\int_{\omega_j}|z_j(t,1,x)|^2dxdt\right).$$

Multiplying (KdVd) by u_j and integrating on time and space, we have

$$\|\underline{u}(s,\cdot)\|^2_{\mathbb{L}^2(\mathcal{T})}+(2\alpha-N)\int_0^s |u_1(t,0)|^2dt+2\sum_{j=1}^N\int_0^s\int_0^{\ell_j} a_j|u_j|^2dxdt$$
$$+\sum_{j=1}^N\int_0^s |\partial_x u_j(t,0)|^2dt+2\sum_{j=1}^N\int_0^s\int_0^{\ell_j} b_ju_j(t-h_j,x)u_j(t,x)dxdt=\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})}.$$

Integrating again over $(0,T)$ this relation, we get,

$$T\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})}\leq\int_0^T \|\underline{u}(t,\cdot)\|^2_{\mathbb{L}^2(\mathcal{T})}dt+T\int_0^T\sum_{j=1}^N|\partial_x u_j(t,0)|^2dt$$
$$+(2\alpha-N)T\int_0^T |u_1(t,0)|^2dt+2T\sum_{j=1}^N\int_0^T\int_0^{\ell_j} a_j|u_j|^2dxdt$$
$$+2T\sum_{j=1}^N\int_0^T\int_0^{\ell_j} b_ju_j(t-h_j,x)u_j(t,x)dxdt.$$

Note now that

$$\int_0^T\int_0^{\ell_j} b_ju_j(t-h_j,x)u_j(t,x)dxdt$$
$$\leq\frac{1}{2}\int_0^T\int_{\omega_j} b_j|u_j|^2dxdt+\frac{1}{2}\int_0^T\int_{\omega_j} b_j|u_j(t-h_j,x)|^2dxdt,$$

and then

$$T\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})}\leq C\left(\int_0^T \|\underline{u}(t,\cdot)\|^2_{\mathbb{L}^2(\mathcal{T})}dt+(2\alpha-N)\int_0^T |u_1(t,0)|^2dt\right.$$
$$+\sum_{j=1}^N\int_0^T |\partial_x u_j(t,0)|^2dt+\sum_{j=1}^N\int_0^T\int_0^{\ell_j} a_j|u_j|^2dxdt$$
$$\left.+\sum_{j=1}^N\int_0^T\int_{\omega_j}|z_j(t,1,x)|^2dxdt\right).$$

Joining the estimates for $\underline{u}^0$ and $\underline{z}^0$ we get

$$\sum_{j=1}^N\int_0^{\ell_j}|u_j^0|^2dx+\sum_{j=1}^N h_j\int_{\omega_j}\int_0^1\xi_j(x)|z_j^0(-h_j,\rho,x)|^2dxd\rho$$
$$\leq C\left(\int_0^T \|\underline{u}(t,\cdot)\|^2_{\mathbb{L}^2(\mathcal{T})}dt+\sum_{j=1}^N\int_0^T |\partial_x u_j(t,0)|^2dt+(2\alpha-N)\int_0^T |u_1(t,0)|^2dt\right.$$
$$\left.+\sum_{j=1}^N\int_0^T\int_0^{\ell_j} a_j|u_j|^2dxdt+\sum_{j=1}^N\int_0^T\int_{\omega_j}|z_j(t,1,x)|^2dxdt\right).$$

This inequality is quite similar to the observability inequality (Obs). Moreover to prove our result it is enough to get that for any T, $R > 0$ there exists $C = C(R,T) > 0$ such that for any solutions of (KdVd) with $\|U_0\|_H \le R$ we have

$$\int_0^T \|\underline{u}(t,\cdot)\|^2_{\mathbb{L}^2(\mathcal{T})} dt \le C\left(\sum_{j=1}^N \int_0^T |\partial_x u_j(t,0)|^2 dt + (2\alpha - N)\int_0^T |u_1(t,0)|^2 dt \right.$$
$$\left. + \sum_{j=1}^N \int_0^T \int_0^{\ell_j} a_j |u_j|^2 dx dt + \sum_{j=1}^N \int_0^T \int_{\omega_j} |z_j(t,1,x)|^2 dx dt \right).$$

Suppose that this inequality does not hold. Then there exists $(\underline{u}^n)_{n\in\mathbb{N}} \subset \mathbb{B}$ solution of (KdVd) with $\|U_0^n\|_H \le R$ such that

$$\lim_{n\to\infty} \frac{\int_0^T \|\underline{u}^n(t,\cdot)\|^2_{\mathbb{L}^2(\mathcal{T})} dt}{\|\partial_x \underline{u}^n(t,0)\|^2_{L^2(0,T)} + (2\alpha - N)\|u_1^n(t,0)\|^2_{L^2(0,T)} + \sum_{j=1}^N \int_0^T \int_0^{\ell_j} a_j |u_j^n|^2 dx dt + \sum_{j=1}^N \int_0^T \int_{\omega_j} |z_j^n(t,1,x)|^2 dx dt} = \infty.$$

Take $\lambda^n = \|\underline{u}^n\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))}$, $\underline{v}^n := \frac{\underline{u}^n}{\lambda^n}$ and $y_j^n(t,\rho,x) = v_j^n\big|_{\omega_j}(t - h_j\rho, x)$ $x \in \omega_j$, $\rho \in (0,1)$. Then, $\underline{v}^n$ satisfies

$$\begin{cases} \partial_t v_j^n(t,x) + \partial_x v_j^n(t,x) + \partial_x^3 v_j^n(t,x) + a_j(x) v_j^n(t,x) & \\ +b_j(x) v_j^n(t-h_j,x) + \lambda^n v_j^n(t,x)\partial_x v_j^n(t,x) = 0, & x \in (0,\ell_j),\ t > 0,\ j = 1,\cdots,N, \\ v_j^n(t,0) = v_k^n(t,0), & \forall j,k = 1,\cdots N, \\ \sum_{j=1}^N \partial_x^2 v_j^n(t,0) = -\alpha v_1^n(t,0) - \lambda^n \dfrac{N}{3}(v_1^n(t,0))^2, & t > 0, \\ v_j^n(t,\ell_j) = \partial_x v_j^n(t,\ell_j) = 0, & t > 0,\ j = 1,\cdots,N, \\ \|\underline{v}^n\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} = 1. & \end{cases} \tag{3.11}$$

and

$$\|\partial_x \underline{v}^n(t,0)\|^2_{L^2(0,T)} + (2\alpha - N)\|v_1^n(t,0)\|^2_{L^2(0,T)} + \sum_{j=1}^N \int_0^T \int_0^{\ell_j} a_j |v_j^n|^2 dx dt$$
$$+ \sum_{j=1}^N \int_0^T \int_{\omega_j} |y_j^n(t,1,x)|^2 dx dt \to 0. \tag{3.12}$$

Now multiplying (3.11) by v_j^n and integrating over $(0,T)\times(0,t)\times(0,\ell_j)$ we can get

$$T\|\underline{v}^n(0,\cdot)\|^2_{\mathbb{L}^2(\mathcal{T})} \le C\left(\int_0^T \|\underline{v}^n(t,\cdot)\|^2_{\mathbb{L}^2(\mathcal{T})} dt + \|\partial_x \underline{v}^n(t,0)\|^2_{L^2(0,T)} + \|\underline{v}^n(-\underline{h}\cdot,\cdot)\|^2_{\mathbb{L}^2(\Omega)}\right.$$
$$\left. + (2\alpha - N)\|v_1^n(t,0)\|^2_{L^2(0,T)}\right).$$

Now for $T > \mathfrak{h}$,

$$\|\underline{v}^n(-\underline{h}\cdot,\cdot)\|^2_{\mathbb{L}^2(\Omega)} = \sum_{j=1}^{N}\int_{\omega_j}\int_0^1 |v_j^n(-h_j\rho,x)|^2 d\rho dx = \sum_{j=1}^{N}\frac{1}{h_j}\int_{\omega_j}\int_{-h_j}^0 |v_j^n(t,x)|^2 dtdx$$
$$\leq \sum_{j=1}^{N}\frac{1}{h_j}\int_{\omega_j}\int_{-h_j}^{T-h_j} |v_j^n(t,x)|^2 dtdx = \sum_{j=1}^{N}\frac{1}{h_j}\int_{\omega_j}\int_0^T |v_j^n(t-h_j,x)|^2 dtdx$$
$$= \sum_{j=1}^{N}\frac{1}{h_j}\int_{\omega_j}\int_0^T |y_j^n(t,1,x)|^2 dtdx.$$

These estimates show us that $(\underline{v}^n(0,\cdot))_{n\in\mathbb{N}}$ is bounded in $\mathbb{L}^2(\mathcal{T})$, also we can see that from the well-posedness of (KdVd) we get

$$\lambda^n = \|\underline{u}^n\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} \leq T\|U_0^n\|_H \leq TR.$$

Consequently in the same sense as (3.8) we can obtain

$$\|\underline{v}^n\|^2_{L^2(0,T;\mathbb{H}^1_e(\mathcal{T}))} \leq C\left(\|U_0^n\|_H^2 + \|U_0^n\|_H^4\right).$$

Thus $(\underline{v}^n)_{n\in\mathbb{N}} \subset L^2(0,T;\mathbb{H}^1_e(\mathcal{T}))$ is bounded and

$$\|v_j^n\partial_x v_j^n\|_{L^2(0,T;L^1(0,\ell_j))} \leq \|\underline{v}^n\|_{C([0,T],\mathbb{L}^2(\mathcal{T}))}\|\underline{v}^n\|_{L^2(0,T;\mathbb{H}^1_e(\mathcal{T}))},$$

what implies that $(v_j^n\partial_x v_j^n)_{n\in\mathbb{N}}$ is subset of $L^2(0,T;L^1(0,\ell_j))$.

With this we can see that $\partial_t v_j^n = -(\partial_x^3 v_j^n + \partial_x v_j^n + \lambda^n v_j^n\partial_x v_j^n + a_j v_j^n + b_j v_j^n(t-h_j))$ is bounded in $L^2(0,T;H^{-2}(0,\ell_j))$ and hence by Aubin-Lions Lemma we can deduce that $(\underline{v}^n)_{n\in\mathbb{N}}$ is relatively compact $L^2(0,T;\mathbb{L}^2(\mathcal{T}))$ and hence we can assume that $\underline{v}^n$ converges strongly at $\underline{v}$ in $L^2(0,T;\mathbb{L}^2(\mathcal{T}))$ with $\|\underline{v}\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} = 1$. Furthermore, passing to the limit on (3.12) we get

$$\|\partial_x\underline{v}(t,0)\|^2_{L^2(0,T)} + (2\alpha-N)\|v_1(t,0)\|^2_{L^2(0,T)} + \sum_{j=1}^{N}\int_0^T\int_{\omega_j}|v_j(t-h_j)|^2 dxdt$$
$$+\sum_{j=1}^{N}\int_0^T\int_0^{\ell_j} a_j|v_j|^2 dxdt \leq \liminf\Bigg(\|\partial_x\underline{v}^n(t,0)\|^2_{L^2(0,T)} + (2\alpha-N)\|v_1^n(t,0)\|^2_{L^2(0,T)}$$
$$+\sum_{j=1}^{N}\int_0^T\int_0^{\ell_j} a_j|v_j^n|^2 dxdt + \sum_{j=1}^{N}\int_0^T\int_{\omega_j}|v_j^n(t-h_j)|^2 dxdt\Bigg).$$

Thus $v_j(t,x) \equiv 0$ in $(-h_j,T)\times\omega_j$ and $(2\alpha-N)v_j(t,0) = \partial_x v_j(t,0) = 0$ in $(0,T)$ for all $j = 1,\cdots,N$. Also as $(\lambda^n)_{n\in\mathbb{N}}$ is bounded, we can extract a convergent subsequence such that $\lambda^n \to \lambda \geq 0$, consequently $\underline{v}$ satisfies $\|\underline{v}\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} = 1$ and the following equation

$$\begin{cases} \partial_t v_j + \partial_x v_j + \partial_x^3 v_j + \lambda v_j \partial_x v_j = 0, & \forall x \in (0,\ell_j),\ t>0,\ j=1,\cdots,N,\\ (2\alpha - N)v_j(t,0) = \partial_x v_j(t,0) = 0, & \forall j = 1,\cdots N,\\ v_j(t,\ell_j) = \partial_x v_j(t,\ell_j) = 0, & \forall j = 1,\cdots N,\\ \sum_{j=1}^N \partial_x^2 v_j(t,0) = -\alpha v_1(t,0) - \lambda \frac{N}{3}(v_1(t,0))^2, & t>0,\\ v_j(t,x) = 0 & (t,x) \in (-h_j,T)\times\omega_j. \end{cases}$$

1. If $\lambda = 0$ the system satisfied by $\underline{v}$ is linear, then thanks Holmgrem's Theorem $\underline{v} = 0$, that contradicts the fact that $\|\underline{v}\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} = 1$.
2. If $\lambda > 0$. In this case we have to prove that $v_j \in L^2(0,T;H^3(0,\ell_j))$ in order to apply Theorem 3.3. Consider $w_j = \partial_t v_t$ then

$$\begin{cases} \partial_t w_j + \partial_x w_j + \partial_x^3 w_j + \lambda w_j \partial_x v_j + \lambda v_j \partial_x w_j = 0, & x \in (0,\ell_j),\ t>0,\ j=1,\cdots,N,\\ (2\alpha - N)w_j(t,0) = \partial_x w_j(t,0) = 0, & \forall j = 1,\cdots N,\\ w_j(t,\ell_j) = \partial_x w_j(t,\ell_j) = 0, & \forall j = 1,\cdots N,\\ \sum_{j=1}^N \partial_x^2 w_j(t,0) = -\alpha w_1(t,0) - \lambda \frac{2N}{3} w_1(t,0)v_1(t,0), & t>0,\\ w_j(t,x) = 0 & (t,x) \in (-h_j,T)\times\omega_j,\\ w_j(0,x) = -v'(0,x) - v'''(0,x) - \lambda v(0,x)v'(0,x), & x \in (0,\ell_j),\ j = 1,\cdots,N. \end{cases}$$

Note that $w_j(0,x) \in H^{-3}(0,\ell_j)$, with Lemma A.2 [3] we can get that $w_j(0,x) \in L^2(0,\ell_j)$ and $w_j \in C([0,T],L^2(0,\ell_l)) \cap L^2(0,T;H^1(0,\ell_j))$. Thus $\partial_x^3 v_j = -(\partial_t v_j - \partial_x v_j - \lambda v_j \partial_x v_j \in L^2(0,T;L^2(0,T))$ that implies $v_j \in L^2(0,T;H^3(0,\ell_j))$. Applying Theorem 3.3 we obtain that $v_j = 0$ for all $j = 1,\cdots,N$ that contradicts the fact that $\|\underline{v}\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} = 1$.

Finally we obtain that (Obs) is valid for a solution (KdVd) with $\|U_0\|_H \le R$. We conclude as in the linear case.

□

Remark 3.5 We can observe that the semi-global character is given by the assumption $\|U_0\|_H \le R$ which is necessary in our proof. Specifically it is used to show that $(\lambda^n)_{n\in\mathbb{N}}$ is bounded. An interesting open problem is the following: Is (KdVd) is globally well-posed and globally exponentially stable?

4 Stabilization when not all damped terms are activated

It is known that to obtain exponential stability of a single KdV equation we only need to add a damped term if the length is critical ($L \in \mathcal{N}$) [17]. In the network case, more precisely in [1] Theorem 3.6, the authors consider damping terms a_j applying on the critical lengths edges except at most on one edge.

Now we will prove Theorem 1.4 following closely Section 6 of [21] and [11]. First note that if (1.8) holds the energy of (KdVd) defined by (1.6) satisfies

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leq& -(2\alpha-N)|u_1(t,0)|^2 - \sum_{j=1}^{N}|\partial_x u_j(t,0)|^2 - 2\sum_{j\in I}\int_{\text{supp }a_j} a_j|u_j|^2dx \\
&+\sum_{j\in I}\int_{\omega_j}(b_j-\xi_j)|u_j(t-h_j,x)|^2dx + \sum_{j\in I}\int_{\omega_j}(b_j+\xi_j)|u_j|^2dx \\
&+\sum_{j\in I^*}\int_{\omega_j}(-2a_j(x)+b_j(x)+\xi_j(x))|u_j(t,x)|^2dx - \sum_{j\in I^*}\int_{(0,\ell_j)/\omega_j} a_j(x)|u_j(t,x)|^2dx \\
&+\sum_{j\in I^*}\int_{\omega_j}(b_j(x)-\xi_j(x))|u_j(t-h_j,x)|^2dx.
\end{aligned}
$$

From the last inequality we can see that in this case the energy of the system (KdVd) is not decreasing in general, this by the action of the terms $b_j+\xi_j>0$ in ω_j for $j\in I$. Following [11] we consider the next auxiliary problem for which the energy will be decreasing. This system is close to (KdVd)

$$
\begin{cases}
\partial_t u_j(t,x)+\partial_x u_j(t,x)+u_j(t,x)\partial_x u_j(t,x)+\partial_x^3 u_j(t,x) & \\
+a_j(x)u_j(t,x)+b_j(x)u_j(t-h_j,x)+\eta b_j(x)u_j(t,x)\mathbb{1}_I(j)=0, & x\in(0,\ell_j),\ t>0,\ j=1,\cdots,N,\\
u_j(t,0)=u_k(t,0), & \forall j,k=1,\cdots N,\\
\sum_{j=1}^N \partial_x^2 u_j(t,0) = -\alpha u_1(t,0)-\frac{N}{3}u_1^2(t,0), & t>0,\\
u_j(t,\ell_j)=\partial_x u_j(t,\ell_j)=0, & t>0,\ j=1,\cdots,N,\\
u_j(0,x)=u_j^0(x), & x\in(0,\ell_j),\\
u_j(t,x)=z_j^0(t,x), & (t,x)\in(-h_j,0)\times(0,\ell_j).
\end{cases}
\tag{Aux}
$$

where $\mathbb{1}_I(j)$ is the indicator function of the set I and $\eta>0$. Then we consider the energy (1.6) with $\xi_j=\eta b_j$ for $j\in I$, that is

$$
\begin{aligned}
E(t) =& \sum_{j=1}^{N}\int_0^{\ell_j}|u_j|^2dx + \eta\sum_{j\in I}h_j\int_{\omega_j}\int_0^1 b_j|u_j(t-h_j\rho,x)|^2dxd\rho \\
&+\sum_{j\in I^*}h_j\int_{\omega_j}\int_0^1 \xi_j|u_j(t-h_j\rho,x)|^2dxd\rho
\end{aligned}
\tag{4.1}
$$

where in this case for all $j\in I^*$, ξ_j is a non-negative function belonging to $L^\infty(0,\ell_j)$ such that supp ξ_j = supp $b_j=\omega_j$ and

$$
b_j(x)+c_0 \leq \xi_j(x) \leq 2a_j(x)-b_j(x)-c_0, \text{ in } \omega_j, \text{ for } j\in I^*. \tag{4.2}
$$

Easy calculations show us that if $\eta>1$, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t) \le& -(2\alpha - N)|u_1(t,0)|^2 - \sum_{j=1}^{N}|\partial_x u_j(t,0)|^2 - 2\sum_{j\in I}\int_{\text{supp }a_j} a_j|u_j|^2dx\\
&+\sum_{j\in I}(1-\eta)\int_{\omega_j} b_j|u_j|^2dx + \sum_{j\in I}(1-\eta)\int_{\omega_j} b_j|u_j(t-h_j)|^2dx\\
&+\sum_{j\in I^*}\int_{\omega_j}(-2a_j(x)+b_j(x)+\xi_j(x))|u_j(t,x)|^2dx - \sum_{j\in I^*}\int_{(0,\ell_j)/\omega_j} a_j(x)|u_j(t,x)|^2dx\\
&+\sum_{j\in I^*}\int_{\omega_j}(b_j(x)-\xi_j(x))|u_j(t-h_j,x)|^2dx \le 0.\end{aligned}$$

The main idea to deal with the case when supp $b_j \not\subset$ supp a_j is to show the exponential stability of the linearization around 0 of (Aux) via a Lyapunov function following Section 3.1 and then pass to (LKdVd) using a perturbation result.

More precisely we are going to use the following theorem.

Theorem 4.1 (Theorem 1.1, [16]) *Let X be a Banach space and let A be a the infinitesimal generator of a C_0 semigroup $T(t)$ on X satisfying $\|T(t)\| \le Me^{\omega t}$. If B is bounded linear operator on X, then $A+B$ is the infinitesimal generator of a C_0 semigroup $S(t)$ on X satisfying $\|S(t)\| \le Me^{(\omega+M\|B\|)t)}$.*

Remark 4.1 As we we said before we use a Lyapunov approach for the auxiliary system, for that we expect that our result holds for $L < \frac{\sqrt{3}}{2}\pi$, $\alpha > n/2$ and small initial data. Also observing Theorem 4.1 we must require that $\|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})}$ is small enough.

We start by proving the well-posedness of the linearization of (Aux) around 0. We omitted the details because they are closely similar to Section 2,

$$\begin{cases}\partial_t u_j(t,x)+\partial_x u_j(t,x)+\partial_x^3 u_j(t,x)+a_j(x)u_j(t,x) & \\
+b_j(x)u_j(t-h_j,x)+\eta b_j(x)u_j(t,x)\mathbb{1}_I(j)=0, & x\in(0,\ell_j),\ t>0,\ j=1,\cdots,N,\\
u_j(t,0)=u_k(t,0), & \forall j,k=1,\cdots N,\\
\sum_{j=1}^N \partial_x^2 u_j(t,0) = -\alpha u_1(t,0), & t>0,\\
u_j(t,\ell_j)=\partial_x u_j(t,\ell_j)=0, & t>0,\ j=1,\cdots,N,\\
u_j(0,x)=u_j^0(x), & x\in(0,\ell_j),\\
u_j(t,x)=z_j^0(t,x), & (t,x)\in(-h_j,0)\times(0,\ell_j).\end{cases} \tag{LAux}$$

We set again $z_j(t,\rho,x) = u_j\big|_{\omega_j}(t-h_j\rho,x)$ $x\in\omega_j$, $\rho\in(0,1)$. Note that in this case as $\xi_j = \eta b_j$ the inner product defined for H in Section 2 becomes

$$\begin{aligned}\left\langle\begin{pmatrix}\underline{u}\\ \underline{z}\end{pmatrix},\begin{pmatrix}\underline{v}\\ \underline{y}\end{pmatrix}\right\rangle_H =& \sum_{j=1}^N\int_0^{\ell_j} u_j(x)v_j(x)dx + \eta\sum_{j\in I}h_j\int_{\omega_j}\int_0^1 b_j(x)z_j(\rho,x)y_j(\rho,x)d\rho dx\\
&+\sum_{j\in I^*}h_j\int_{\omega_j}\int_0^1 \xi_j(x)z_j(\rho,x)y_j(\rho,x)d\rho.\end{aligned}$$

Then (LAux) can be written as

$$\begin{cases} U_t(t) = \mathcal{A}_0 U(t),\ t > 0 \\ \quad U(0) = U_0. \end{cases} \tag{4.3}$$

where, $U = \begin{pmatrix} \underline{u} \\ \underline{z} \end{pmatrix}$, $U_0 = \begin{pmatrix} \underline{u}^0 \\ \underline{z}^0\big|_{\omega}(-\underline{h}\cdot,\cdot) \end{pmatrix}$ and the operator $\mathcal{A}_0$ is defined by:

$$\mathcal{A}_0 U = \begin{pmatrix} -(\mathrm{D}_x(\mathcal{T}) + \mathrm{D}_x^3(\mathcal{T}))\underline{u} - \underline{a}.*\underline{u} - \underline{b}.*\underline{\tilde{z}}(1,\cdot) - \eta \underline{b}^I.*\underline{u} \\ -\dfrac{1}{\underline{h}}.*\mathrm{D}_\rho(\mathcal{T})\underline{z} \end{pmatrix}$$

in which

$$(\underline{b}^I)_j = \begin{cases} b_j,\ \ j \in I, \\ 0,\ \ j \in I^*. \end{cases}$$

and $D(\mathcal{A}_0) = D(\mathcal{A})$.

Theorem 4.2 *Assume $\underline{a}$, $\underline{b} \in \mathbb{L}^\infty(\mathcal{T})$ componentwise non-negative that satisfy* (1.1) *and* (1.9)*. Let $U_0 \in H$ and $\eta > 1$. Then there exist a unique solution $U \in C([0,\infty);H)$ of* (4.3)*. Moreover if $U_0 \in D(\mathcal{A})$ then U is a classical solution and*

$$U \in C([0,\infty); D(\mathcal{A}_0)) \cap C^1([0,\infty); H).$$

Proof Let $U = \begin{pmatrix} \underline{u} \\ \underline{z} \end{pmatrix} \in D(\mathcal{A}_0)$, then

$$\begin{aligned}
\langle \mathcal{A}_0 U, U\rangle \leq \left(\frac{N}{2} - \alpha\right) u_1^2(0) - \frac{1}{2}\sum_{j=1}^{N}(\partial_x u_j(0))^2 - \sum_{j\in I}\int_{\text{supp } a_j} a_j|u_j|^2 dx \\
+\frac{1}{2}\sum_{j\in I}(1-\eta)\int_\omega b_j|u_j|^2 dx + \frac{1}{2}\sum_{j\in I}(1-\eta)\int_\omega b_j|u_j(t-h_j)|^2 dx \\
-\frac{1}{2}\sum_{j\in I^*}\int_{(0,\ell_j)/\omega_j} a_j(x)|u_j(t,x)|^2 dx + \sum_{j\in I^*}\int_{\omega_j}(-a_j(x) + \frac{b_j(x)}{2} + \frac{\xi_j(x)}{2})|u_j(t,x)|^2 dx \\
+\frac{1}{2}\sum_{j\in I^*}\int_{\omega_j}(b_j(x) - \xi_j(x))|u_j(t-h_j,x)|^2 dx \leq 0.
\end{aligned}$$

thus $\mathcal{A}_0$ is dissipative. Moreover

$$\mathcal{A}_0^*\begin{pmatrix} \underline{v} \\ \underline{y} \end{pmatrix} = \begin{pmatrix} (\mathrm{D}_x(\mathcal{T}) + \mathrm{D}_x^3(\mathcal{T}))\underline{v} - \underline{a}.*\underline{v} + \eta\underline{b}.*\underline{\tilde{y}}(0,\cdot) - \eta\underline{b}^I.*\underline{v} \\ \dfrac{1}{\underline{h}}.*\mathrm{D}_\rho(\mathcal{T})\underline{y} \end{pmatrix}$$

$$\begin{aligned}
D(\mathcal{A}_0^*) = \Bigg\{ \begin{pmatrix} \underline{v} \\ \underline{y} \end{pmatrix}, \underline{v} \in \left(\prod_{j=1}^{N} H^3(0,\ell_j)\right) \cap \mathbb{H}_e^1(\mathcal{T}), \sum_{j=1}^{N}\frac{\mathrm{d}^2 v_j}{\mathrm{d}x^2}(0) = (\alpha - N)v_1(0), \\
\partial_x v_j(0) = 0,\ \forall j = 1,\cdots,N,\ \underline{y} \in \prod_{j=1}^{N} L^2(H^1(0,1)\times w_j),\ y_j(1,x) = -\frac{1}{\eta}v_j\big|_{\omega_j}(x) \\
\text{for } j \in I \ \text{ and } y_j(1,x) = -\frac{b_j}{\xi_j}v_j\big|_{\omega_j}(x) \text{ for } j \in I^* \Bigg\}.
\end{aligned}$$

Let $V = \begin{pmatrix} v \\ \underline{y} \end{pmatrix} \in D(\mathcal{A}_0^*)$, then

$$\begin{aligned}\langle \mathcal{A}_0^* V, V\rangle \leq & -\frac{1}{2}\sum_{j=1}^{N}|\partial_x v_j(\ell_j)|^2 + \left(\frac{N}{2}-\alpha\right)v_1^2(0) - \sum_{j=1}^{N}\int_{\text{supp } a_j} a_j|v_j|^2 dx \\ & + \sum_{j=1}^{N}\int_{\omega_j}\left(-\frac{\eta}{2}+\frac{2}{2\eta}\right)b_j(x)v_j^2(x)dx + \sum_{j\in I^*}\int_{\omega_j}\left(-a_j+\frac{\xi_j}{2}+\frac{b_j^2}{2\xi_j}\right)|v_j|^2 dx \\ & - \sum_{j\in I^*}\int_{(0,\ell_j)\setminus\omega_j} a_j|v_j|^2 dx - \frac{1}{2}\sum_{j\in I^*}\int_{\omega_j}\xi_j|y_j(0,x)|^2 dx \leq 0\end{aligned}$$

thus $\mathcal{A}_0^*$ is dissipative. □

Now to prove the exponential stability of (LAux) we consider the following Lyapunov function:

$$V(t) = E(t) + \mu_1 V_1 + \mu_2 V_2 \tag{4.4}$$

where $\mu_1, \mu_2 > 0$, $E(t)$ is defined by (4.1), $V_1(t)$ defined in (3.1) and $V_2(t)$ is given by

$$\begin{aligned}V_2(t) = & \sum_{j\in I} h_j \int_{\omega_j}\int_0^1 (1-\rho)b_j(x)|u_j(t-h_j\rho,x)|^2 dx d\rho \\ & + \sum_{j\in I^*} h_j \int_{\omega_j}\int_0^1 (1-\rho)|u_j(t-h_j\rho,x)|^2 dx d\rho.\end{aligned}$$

Proposition 4.1 *Assume $\underline{a}$, $\underline{b} \in \mathbb{L}^\infty(\mathcal{T})$ componentwise non-negative that satisfy* (1.1) *and* (1.9). *Let $\alpha > \frac{N}{2}$, $\eta > 1$ and $(\ell_j)_{j=1}^N \subset (0,+\infty)$ such that $L < \frac{\sqrt{3}}{2}\pi$. Then for every $U_0 \in H$, the energy of* (LAux) *defined by* (4.1) *decays exponentially, that is, there exists $C > 0$, $\gamma > 0$ such that*

$$E(t) \leq CE(0)e^{-2\gamma t}.$$

where

$$\begin{aligned}\gamma \leq \min\left\{\frac{(3\mu_1\pi - \mu_1 4L^2)}{8L^2(1+L\mu_1)}, \min_{j\in I}\frac{\mu_2}{2h_j(\eta+\mu_2)}, \min_{j\in I^*}\frac{\mu_2}{2h_j(\xi_j+\mu_2)}\right\}, \\ C = \left(1+\max\left\{L\mu_1, \frac{\mu_2}{\eta}, \frac{\mu_2}{b_0}\right\}\right).\end{aligned} \tag{4.5}$$

for μ_1 and μ_2 such that

$$0 < \mu_1 < \left\{1, \frac{\eta-1}{L}, \frac{1}{N}(2\alpha - N), \min_{j\in I^*}\left\{\inf_{\omega_j}\frac{\xi_j - b_j}{Lb_j}, \inf_{\omega_j}\frac{2a_j - b_j - \xi_j}{Lb_j}\right\}\right\}$$

$$0 < \mu_2 < \min\left\{\eta - 1 - L\mu_1, \min_{j\in I^*} 2a_j - b_j - \xi_j - L\mu_1 b_j\right\}.$$

Proof Let $\underline{u}$ be a regular solution of (LAux) with $U_0 \in D(\mathcal{A}_0)$. Clearly with this definition of $V(t)$ we have that

$$E(t) \leq V(t) \leq \left(1 + \max\left\{L\mu_1, \frac{\mu_2}{\eta}, \frac{\mu_2}{b_0}\right\}\right) E(t).$$

Now, integrating by parts we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leq{}& (N-2\alpha)\,|u_1(t,0)|^2 - \sum_{j=1}^{N} |\partial_x u_j(t,0)|^2 - 2\sum_{j\in I}\int_{\text{supp } a_j} a_j|u_j|^2 dx \\
&+(1-\eta)\sum_{j\in I}\int_{\omega_j} b_j|u_j|^2 dx + (1-\eta)\sum_{j\in I}\int_{\omega_j} b_j|u_j(t-h_j,x)|^2 dx \\
&+\sum_{j\in I^*}\int_{\omega_j}(-2a_j+b_j+\xi_j)|u_j|^2 dx - \sum_{j\in I^*}\int_{\text{supp } a_j\setminus\omega_j} a_j|u_j|^2 dx \\
&+\sum_{j\in I^*}\int_{\omega_j}(b_j-\xi_j)|u_j(t-h_j,x)|^2 dx,
\end{aligned}$$

together with

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V_1(t) ={}& \sum_{j=1}^{N}\int_0^{\ell_j}|u_j|^2 dx - 3\sum_{j=1}^{N}\int_0^{\ell_j}|\partial_x u_j|^2 dx - 2\sum_{j=1}^{N} u_1(t,0)\partial_x u_j(t,0) \\
&-2\eta\sum_{j\in I}\int_{\omega_j} x b_j|u_j|^2 dx - 2\sum_{j=1}^{N}\int_{\text{supp } a_j} x a_j|u_j|^2 dx - 2\sum_{j=1}^{N}\int_{\omega_j} x b_j u_j(t,x)u_j(t-h_j,x)dx, \\
\frac{\mathrm{d}}{\mathrm{d}t}V_2(t) ={}& \sum_{j\in I}\int_{\omega_j} b_j|u_j|^2 dx + \sum_{j\in I^*}\int_{\omega_j}|u_j|^2 dx - \sum_{j\in I}\int_{\omega_j} b_j\int_0^1|u_j(t-h_j\rho,x)|^2 d\rho dx \\
&-\sum_{j\in I^*}\int_{\omega_j}\int_0^1|u_j(t-h_j\rho,x)|^2 d\rho dx.
\end{aligned}$$

Using integrations by parts and Poincaré's inequality, we can easily check that for $\gamma > 0$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V(t) + 2\gamma V(t) \leq{}& (N-2\alpha+\mu_1 N)\,|u_1(t,0)|^2 + (\mu_1 - 1)\sum_{j=1}^{N}|\partial_x u_j(t,0)|^2 \\
&+\sum_{j\in I}\int_{\omega_j} b_j(1-\eta+\mu_2+\mu_1 L)|u_j|^2 dx + \sum_{j\in I}\int_{\omega_j} b_j(1-\eta+\mu_1 L)|u_j(t-h_j,x)|^2 dx \\
&+\sum_{j\in I^*}\int_{\omega_j}(-2a_j+b_j+\xi_j+\mu_2+L\mu_1 b_j)|u_j|^2 dx
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j\in I^*}\int_{\omega_j}(b_j-\xi_j+\mu_1 Lb_j)|u_j(t-h_j)|^2dx\\
&+\sum_{j\in I}\int_{\omega_j}\int_0^1(2\gamma h_j(\mu_2+\eta)-\mu_2)|u_j(t-h_j\rho,x)|^2dx\\
&+\sum_{j\in I^*}\int_{\omega_j}\int_0^1(2\gamma h_j(\mu_2+\xi_j)-\mu_2)|u_j(t-h_j\rho,x)|^2dx\\
&+\left[\frac{4L^2(\mu_1+2\mu_1\gamma L+2\gamma)}{\pi^2}-3\mu_1\right]\sum_{j=1}^N\int_0^{\ell_j}|\partial_x u_j(t,x)|^2dx.
\end{aligned}
$$

Taking

$$0<\mu_1<\left\{1,\frac{\eta-1}{L},\frac{1}{N}(2\alpha-N),\min_{j\in I^*}\left\{\inf_{\omega_j}\frac{\xi_j-b_j}{Lb_j},\inf_{\omega_j}\frac{2a_j-b_j-\xi_j}{Lb_j}\right\}\right\}$$

$$0<\mu_2<\min\left\{\eta-1-L\mu_1,\min_{j\in I^*}2a_j-b_j-\xi_j-L\mu_1 b_j\right\},$$

and using that $L>\frac{\sqrt{3}}{2}\pi$ we can take

$$\gamma\le\min\left\{\frac{(3\mu_1\pi-\mu_1 4L^2)}{8L^2(1+L\mu_1)},\min_{j\in I}\frac{\mu_2}{2h_j(\eta+\mu_2)},\min_{j\in I^*}\frac{\mu_2}{2h_j(\xi_j+\mu_2)}\right\}.$$

With this $\dfrac{\mathrm{d}}{\mathrm{d}t}V(t)+2\gamma V(t)\le 0$ which implies

$$E(t)\le\left(1+\max\left\{L\mu_1,\frac{\mu_2}{\eta},\frac{\mu_2}{b_0}\right\}\right)E_0e^{-2\gamma t}.$$

By density we can extend the result to $U_0\in H$. □

Now we will obtain a stability result of (LKdVd) using a perturbation argument. Note first that the operator $\mathcal{A}$ introduced in Section 2 and associated with (LKdVd) can be written as

$$\mathcal{A}=\mathcal{A}_0+\mathcal{B},$$

where $D(\mathcal{A})=D(\mathcal{A}_0)$ and $\mathcal{B}$ is the bounded operator on H defined by

$$\mathcal{B}U=\begin{pmatrix}\eta\underline{b}^I.*\underline{u}\\ \underline{0}\end{pmatrix},\quad U=\begin{pmatrix}\underline{u}\\ \underline{z}\end{pmatrix}\in H.$$

Proposition 4.2 *Assume $\underline{a}$, $\underline{b}\in\mathbb{L}^\infty(\mathcal{T})$ componentwise non-negative that satisfy* (1.1) *and* (1.9). *Let $\alpha>\frac{N}{2}$, $\eta>1$ and $(\ell_j)_{j=1}^N\subset(0,+\infty)$ such that $L<\frac{\sqrt{3}}{2}\pi$, then for every $U_0\in H$ there exists a unique mild solution $U\in C([0,\infty),H)$ for* (LKdVd). *Moreover if $U_0\in D(\mathcal{A})$ then the solution is classical and $U\in C([0,\infty);D(\mathcal{A}))\cap C^1([0,\infty),H)$. Furthermore there exists $\delta=\delta(\alpha,\eta,L,\underline{h})>0$ such that if*

$$\|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})}\le\delta,$$

then for every $U_0 \in H$*, the solution of* (LKdVd) *satisfies*

$$E(t) \leq CE(0)e^{-\gamma t}, \quad t > 0,$$

for $C, \gamma > 0$ *defined in Proposition 4.1.*

Proof It is enough to apply Theorem 4.1. We note that $\|\mathcal{B}\| \leq \eta \|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})}$ and then remark that

$$-\frac{\gamma}{2} + \sqrt{C}\eta \|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})} < 0 \Longleftrightarrow \|\underline{b}\|_{\mathbb{L}^\infty(\mathcal{T})} < \frac{\gamma}{2\eta\sqrt{C}}.$$

□

Finally, we obtain the local exponential stability for (KdVd) in the case when $\operatorname{supp} b_j \not\subset \operatorname{supp} a_j$, for $j \in I^* \subset \{1, \cdots, N\}$ stated in Theorem 1.4.

Proof of Theorem 1.4 :
We just adapt the proof of Theorem 2.2 and Theorem 1.2 to obtain the exponential stability of the nonlinear case using the stability of (LAux) and small initial data. □

5 Numerical Simulations

The purpose of this section is to illustrate the stabilization results obtained in this work. For that we are going to present some numerical simulations adapting the schemes used in [2, 5, 14]. We choose a final time T and for simplicity we take $\ell_j = L$ and a_j, b_j constant on their support for all $j = 1, \cdots, N$. We build a uniform spatial and time discretization of $N_x + 1$ and $N_t + 1$ points respectively, separated by the steps $\Delta x = L/N_x$ and $\Delta t = T/N_t$. To deal with the delay term we choose the delay step $\Delta\rho = 1/N_\rho$. Now we introduce the notation $u_j(n\Delta t, i\Delta x) = u^n_{j,i}$ and $z_j(n\Delta t, k\Delta\rho, i\Delta x) = z^n_{j,i,k}$ for $i = 0, \cdots, N_x$, $k = 0, \cdots, N_\rho$ and $n = 0, \cdots, N_t$. We use the following approximation for the derivatives:

$$\mathrm{D}^+_x y_i = \frac{y_{i+1} - y_i}{\Delta x}, \quad \mathrm{D}^-_x y_i = \frac{y_i - y_{i-1}}{\Delta x}, \quad \mathrm{D}_x y_i = \frac{y_{i+1} - y_{i-1}}{2\Delta x}, \quad \mathrm{D}^+_\rho e_k = \frac{e_{k+1} - e_k}{\Delta\rho}.$$

In order to approximate the term of third order ∂^3_x we use $\mathrm{D}^+_x\mathrm{D}^+_x\mathrm{D}^-_x$. Now, to consider the nonlinear terms we use explicit approximation $y^n_i \mathrm{D}^+_x y^n_i$ and for the nonlinear boundary condition we use a forward approximation for the second derivative which gives

$$\left(\frac{1}{(\Delta x)^2} + \alpha\right) u^{n+1}_{j,0} - \frac{2}{(\Delta x)^2} u^{n+1}_{j,1} + \frac{1}{(\Delta x)^3} u^{n+1}_{j,2} = -\frac{1}{3}(u^n_{j,0})^2, \quad j = 1, \cdots, N, \; n = 1 \cdots, N_t.$$

Note now that by the boundary conditions we have that $u^n_{j,N_x} = u^n_{j,N_x-1} = 0$, $u^n_{j,0} = u^n_k$ for all $n = 0, \cdot, N_j$ and $j, k = 1 \cdots, N$. Now we define I_{ω_j}, the set of index such that $i \in I_{\omega_j}$ if $i\Delta x \in \omega_j$. Then taking $C = \mathrm{D}^+_x\mathrm{D}^+_x\mathrm{D}^-_x + \mathrm{D}_x$ our scheme can be seen as

$$
\begin{cases}
\left(\frac{1}{(\Delta x)^2}+\alpha\right)u_{j,0}^{n+1}-\frac{2}{(\Delta x)^2}u_{j,1}^{n+1}+\frac{1}{(\Delta x)^3}u_{j,2}^{n+1}=-\frac{1}{3}(u_{j,0}^n)^2, & j=1,\cdots,N,\ n=1\cdots,N_t,\\[2ex]
\frac{u_{j,i}^{n+1}-u_{j,i}^n}{\Delta t}+(Cu_j^{n+1})_i+a_ju_j^{n+1}+b_jz_{j,i,N_\rho}+u_{j,i}^n\mathrm{D}_x^+u_{j,i}^n=0, & i\in I_{\omega_j},\ i\neq 0,\ j=1,\cdots,N,\\
\frac{u_{j,i}^{n+1}-u_{j,i}^n}{\Delta t}+(Cu_j^{n+1})_i+u_{j,i}^n\mathrm{D}_x^+u_{j,i}^n=0, & i\in\{1,\cdots,N\}\backslash I_{\omega_j},\ i\neq 0,\\
 & j=1,\cdots,N,\\
h_j\frac{z_{j,i,k}^{n+1}-z_{j,i,k}^n}{\Delta t}+(\mathrm{D}_\rho^+z_{j,i}^{n+1})_k=0, & k=1,\cdots,N_\rho,\\
u_{j,N_x}^n=u_{j,N_x-1}^n=0 & j=1,\cdots,N,\\
u_{j,0}^n=u_k^n & j,k=1,\cdots,N,\\
z_{j,i,0}^n=u_{j,i}^n, & i\in I_{\omega_j},\ j=1,\cdots,N,\\
u_{j,i}^0=u_j^0(i\Delta x) & i=1,\cdots,N_x,\ j=1,\cdots,N,\\
z_{j,i,k}^0=z_j^0(k\Delta\rho,i\Delta x), & k=1,\cdots,N_\rho,\ i\in I_{\omega_j},\ j=1,\cdots,N.
\end{cases}
\tag{5.1}
$$

Now we use this scheme with the following parameters, $N=4$, $L=2$ and $\alpha=3$, for the discretization we use $N_x=100$, $N_\rho=100$, the initial conditions are $u_j^0=(1-\cos(2\pi x/L))$ and $z_j^0=(1-\cos(2\pi x/L))\cos(2\pi\rho h_j)$. As we say before we consider that the feedback terms are constant on their support, and we take $\omega_1=(0,L/2)$, $\omega_2=(0,L/4)$, $\omega_3=(0,L/2)$ and $\omega_4=(0,L/4)$.

For Figure 2 we use $T=5$, $N_t=100$ and delay $h_1=1$, $h_2=0.5$, $h_3=1$ and $h_4=1$. We can see that when there is not feedback terms ($a_j=b_j=0$), the energy is exponentially decreasing and if we only activate the feedback term without delay, the energy decays more quickly. If we activate both feedback terms with and without delay the energy still decrease exponentially but slowly. Similar case happen if we not activate a feedback term without delay but we consider a feedback term with a small delay ($a_4=0$, $b_j=0.5$). Finally if we consider only the action of delay feedback terms we can observe that in this case the energy decays very slowly.

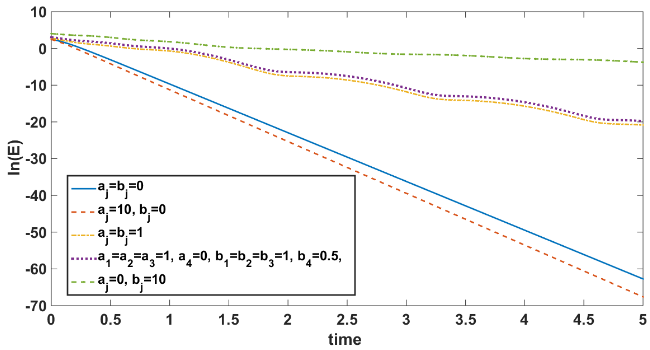

Fig. 2 Time-evolution of $t\mapsto\ln(E(t))$ for different values of feedback terms.

For Figure 3 we use $T = 10$, $N_t = 200$, $a_j = 0$ for $j = 1, \cdots 4$. In this figure we can observe that in the case $a_j = 0$ the energy decays exponentially if the feedback terms with delay are small enough. Also we can see that if the delay is bigger the feedback term with delay has to be smaller as written in Theorem 1.4.

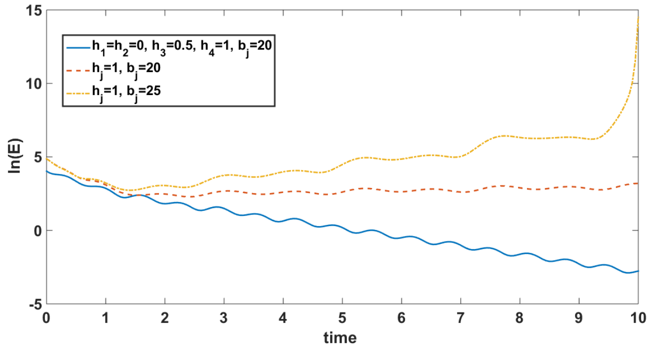

Fig. 3 Time-evolution of $t \mapsto \ln(E(t))$ for different values of feedback with delay term.

For Figure 4 we consider $T = 2$, $N_t = 100$, delays $h_1 = 0.1$, $h_2 = 0.2$, $h_3 = 0.3$ and $h_4 = 0.4$, feedback terms $a_1 = 2$, $a_2 = 4$, $a_3 = 6$, $a_4 = 8$, $b_1 = 0.5$, $b_2 = 1.5$, $b_3 = 2.5$ and $b_4 = 3.5$. We show $\ln(E(t))$ for different initial conditions $u_j^0 = \theta(1 - \cos(2\pi x/L))$ and $z_j^0 = \theta(1 - \cos(2\pi x/L))\cos(2\pi\rho h_j)$, for $\theta = 0.5, 2, 5$. We show the graphics of $\ln\left(\frac{E(t)}{E(0)}\right)$ in order to normalize the energy. Here we can see that the decay rate does not seem to depend of the initial data.

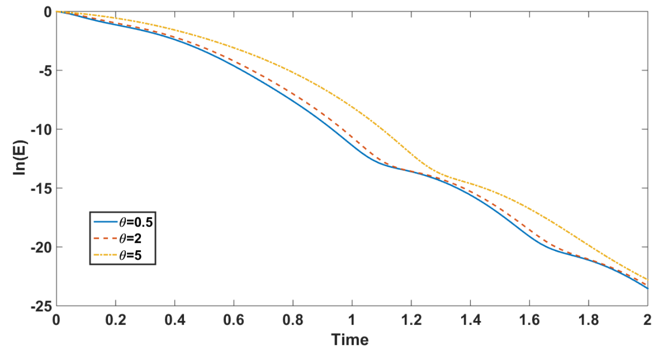

Fig. 4 Time-evolution of $t \mapsto \ln\left(\frac{E(t)}{E(0)}\right)$ for different values of $\theta > 0$.

To end we consider $T = 2$, $N_t = 100$, delays $h_1 = 0.1$, $h_2 = 0.2$, $h_3 = 0.3$ and $h_4 = 0.4$, feedback terms $a_1 = 2$, $a_2 = 4$, $a_3 = 6$, $a_4 = 8$, $b_1 = 0.5$, $b_2 = 1.5$, $b_3 = 2.5$ and $b_4 = 3.5$. We calculate the theoretical decay rate given by Theorem 1.1, we get $\ln(E(t)) \leq 2.9283 - 1.2143 \cdot 10^{-17}t$. Now we make a linear regression for the numerical obtained data and we get $\ln(E(t)) \approx 4.1129 - 12.766t$. From here we can see that the theoretical decay rate given by Theorem 1.1 is much smaller than the one obtained numerically.

6 Conclusions

In this paper, was studied the well-posedness and exponential stability of a KdV equation on a Star Shaped Network with internal delayed feedback terms. The well-posedness was addressed including a new variable in order to take into account the delay and then studying the linearization around 0 of our system we obtain the local well-posedness for the nonlinear equation using the Banach fixed-point theorem.

First was considered the case where the support of delayed terms b_j are included in the support of the feedback terms without delay a_j. In this was possible obtain the local exponential stability using a Lyapunov functional, this result holds for restricted lengths $L < \frac{\sqrt{3}}{2}\pi$, $\alpha > \frac{N}{2}$ and gives us an estimation of the decay rate. This estimation of the decay rate depends strongly on the Lyapunov Function used. Secondly using a contradiction argument an observability inequality for the linear system was derived that gives the exponential stability of the non linear system without restrictions on the lengths and $\alpha \geq \frac{N}{2}$. On a similar way working directly with the nonlinear system a semi-global stabilization result was obtained.

In the last stabilization results the case where non necessarily the support of delayed terms b_j are included in the support of the feedback terms without delay a_j has been considered. If this is the case and if the feedback delayed term b_j is small enough, the local exponential stability for $L < \frac{\sqrt{3}}{2}\pi$ and $\alpha > \frac{N}{2}$ has been obtained.

Finally some numerical simulations have been presented. We showed how feedback delayed terms affects the stability (see Figure 2 and Figure 3). Also we observe that numerically in the case $a_j = 0$ if the feedback delayed terms b_j is big enough the system becomes unstable. Besides, we showed that the decay rate given by Theorem 1.1 is smaller than those obtained in simulations.

To conclude we present some open questions to be investigated:

1. As was said in Remark 3.1 the restriction on L comes from the multiplier x in V_1. Finding a new multiplier in order to obtain a result less restrictive is an open problem.
2. In this paper was considered that the delay acts internally. We are working on a delay term acting on the central node.

3. The tools used in this work are inspired by [9]. For that a future research could be to study a KdV equation with a saturated control on a Star Shaped Network.
4. Typically the KdV equation is globally well-posedness in a bounded domain, the main difficulty to reach global well-posedness in the network case is Proposition 2.6. A global well-posedness of the KdV equation in a Star Shaped Network is an open problem.
5. In [7] a stabilization problem for the linear Kuramoto-Sivashinsky with delayed boundary control was studied. Studying a Kuramoto-Sivashinky equation on networks with or without delay is also a possible future work.

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